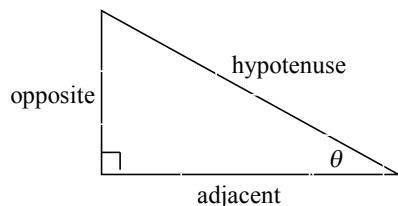


# Trigonometry Definitions and Formulae

### Right triangle definition

For this definition we assume that

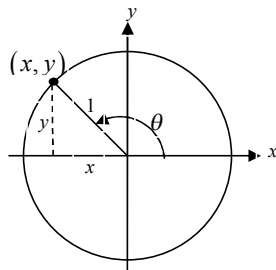
$$0 < \theta < \frac{\pi}{2} \text{ or } 0^\circ < \theta < 90^\circ.$$



$$\begin{aligned} \sin \theta &= \frac{\text{opposite}}{\text{hypotenuse}} & \csc \theta &= \frac{\text{hypotenuse}}{\text{opposite}} \\ \cos \theta &= \frac{\text{adjacent}}{\text{hypotenuse}} & \sec \theta &= \frac{\text{hypotenuse}}{\text{adjacent}} \\ \tan \theta &= \frac{\text{opposite}}{\text{adjacent}} & \cot \theta &= \frac{\text{adjacent}}{\text{opposite}} \end{aligned}$$

### Unit circle definition

For this definition  $\theta$  is any angle.



$$\begin{aligned} \sin \theta &= \frac{y}{1} = y & \csc \theta &= \frac{1}{y} \\ \cos \theta &= \frac{x}{1} = x & \sec \theta &= \frac{1}{x} \\ \tan \theta &= \frac{y}{x} & \cot \theta &= \frac{x}{y} \end{aligned}$$

### Facts and Properties

#### Domain

The domain is all the values of  $\theta$  that can be plugged into the function.

$$\begin{aligned} \sin \theta, \quad \theta &\text{ can be any angle} \\ \cos \theta, \quad \theta &\text{ can be any angle} \\ \tan \theta, \quad \theta &\neq \left(n + \frac{1}{2}\right)\pi, \quad n = 0, \pm 1, \pm 2, \dots \\ \csc \theta, \quad \theta &\neq n\pi, \quad n = 0, \pm 1, \pm 2, \dots \\ \sec \theta, \quad \theta &\neq \left(n + \frac{1}{2}\right)\pi, \quad n = 0, \pm 1, \pm 2, \dots \\ \cot \theta, \quad \theta &\neq n\pi, \quad n = 0, \pm 1, \pm 2, \dots \end{aligned}$$

#### Range

The range is all possible values to get out of the function.

$$\begin{aligned} -1 \leq \sin \theta \leq 1 & \quad \csc \theta \geq 1 \text{ and } \csc \theta \leq -1 \\ -1 \leq \cos \theta \leq 1 & \quad \sec \theta \geq 1 \text{ and } \sec \theta \leq -1 \\ -\infty < \tan \theta < \infty & \quad -\infty < \cot \theta < \infty \end{aligned}$$

#### Period

The period of a function is the number,  $T$ , such that  $f(\theta + T) = f(\theta)$ . So, if  $\omega$  is a fixed number and  $\theta$  is any angle we have the following periods.

$$\begin{aligned} \sin(\omega\theta) &\rightarrow T = \frac{2\pi}{\omega} \\ \cos(\omega\theta) &\rightarrow T = \frac{2\pi}{\omega} \\ \tan(\omega\theta) &\rightarrow T = \frac{\pi}{\omega} \\ \csc(\omega\theta) &\rightarrow T = \frac{2\pi}{\omega} \\ \sec(\omega\theta) &\rightarrow T = \frac{2\pi}{\omega} \\ \cot(\omega\theta) &\rightarrow T = \frac{\pi}{\omega} \end{aligned}$$

### Formulas and Identities

#### Tangent and Cotangent Identities

$$\tan \theta = \frac{\sin \theta}{\cos \theta} \quad \cot \theta = \frac{\cos \theta}{\sin \theta}$$

#### Reciprocal Identities

$$\begin{aligned} \csc \theta &= \frac{1}{\sin \theta} & \sin \theta &= \frac{1}{\csc \theta} \\ \sec \theta &= \frac{1}{\cos \theta} & \cos \theta &= \frac{1}{\sec \theta} \\ \cot \theta &= \frac{1}{\tan \theta} & \tan \theta &= \frac{1}{\cot \theta} \end{aligned}$$

#### Pythagorean Identities

$$\begin{aligned} \sin^2 \theta + \cos^2 \theta &= 1 \\ \tan^2 \theta + 1 &= \sec^2 \theta \\ 1 + \cot^2 \theta &= \csc^2 \theta \end{aligned}$$

#### Even/Odd Formulas

$$\begin{aligned} \sin(-\theta) &= -\sin \theta & \csc(-\theta) &= -\csc \theta \\ \cos(-\theta) &= \cos \theta & \sec(-\theta) &= \sec \theta \\ \tan(-\theta) &= -\tan \theta & \cot(-\theta) &= -\cot \theta \end{aligned}$$

#### Periodic Formulas

If  $n$  is an integer.

$$\begin{aligned} \sin(\theta + 2\pi n) &= \sin \theta & \csc(\theta + 2\pi n) &= \csc \theta \\ \cos(\theta + 2\pi n) &= \cos \theta & \sec(\theta + 2\pi n) &= \sec \theta \\ \tan(\theta + \pi n) &= \tan \theta & \cot(\theta + \pi n) &= \cot \theta \end{aligned}$$

#### Double Angle Formulas

$$\begin{aligned} \sin(2\theta) &= 2 \sin \theta \cos \theta \\ \cos(2\theta) &= \cos^2 \theta - \sin^2 \theta \\ &= 2 \cos^2 \theta - 1 \\ &= 1 - 2 \sin^2 \theta \\ \tan(2\theta) &= \frac{2 \tan \theta}{1 - \tan^2 \theta} \end{aligned}$$

#### Degrees to Radians Formulas

If  $x$  is an angle in degrees and  $t$  is an angle in radians then

$$\frac{\pi}{180} = \frac{t}{x} \Rightarrow t = \frac{\pi x}{180} \quad \text{and} \quad x = \frac{180t}{\pi}$$

#### Half Angle Formulas (alternate form)

$$\begin{aligned} \sin \frac{\theta}{2} &= \pm \sqrt{\frac{1 - \cos \theta}{2}} & \sin^2 \theta &= \frac{1}{2}(1 - \cos(2\theta)) \\ \cos \frac{\theta}{2} &= \pm \sqrt{\frac{1 + \cos \theta}{2}} & \cos^2 \theta &= \frac{1}{2}(1 + \cos(2\theta)) \\ \tan \frac{\theta}{2} &= \pm \sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}} & \tan^2 \theta &= \frac{1 - \cos(2\theta)}{1 + \cos(2\theta)} \end{aligned}$$

#### Sum and Difference Formulas

$$\begin{aligned} \sin(\alpha \pm \beta) &= \sin \alpha \cos \beta \pm \cos \alpha \sin \beta \\ \cos(\alpha \pm \beta) &= \cos \alpha \cos \beta \mp \sin \alpha \sin \beta \\ \tan(\alpha \pm \beta) &= \frac{\tan \alpha \pm \tan \beta}{1 \mp \tan \alpha \tan \beta} \end{aligned}$$

#### Product to Sum Formulas

$$\begin{aligned} \sin \alpha \sin \beta &= \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)] \\ \cos \alpha \cos \beta &= \frac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)] \\ \sin \alpha \cos \beta &= \frac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)] \\ \cos \alpha \sin \beta &= \frac{1}{2}[\sin(\alpha + \beta) - \sin(\alpha - \beta)] \end{aligned}$$

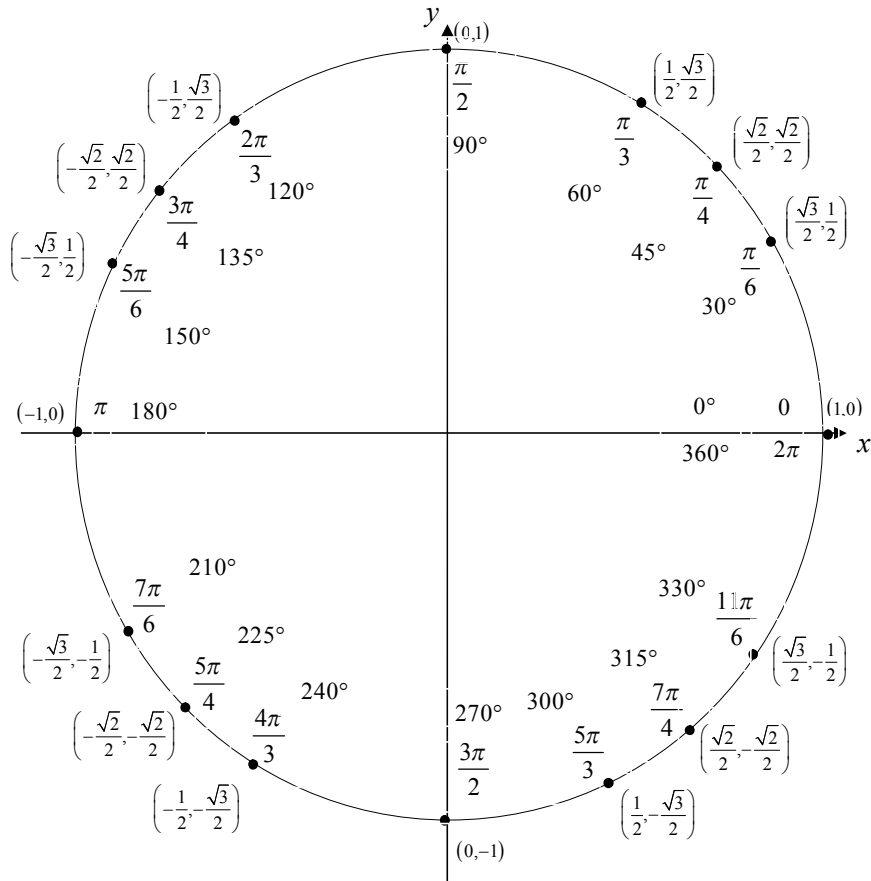
#### Sum to Product Formulas

$$\begin{aligned} \sin \alpha + \sin \beta &= 2 \sin\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right) \\ \sin \alpha - \sin \beta &= 2 \cos\left(\frac{\alpha + \beta}{2}\right) \sin\left(\frac{\alpha - \beta}{2}\right) \\ \cos \alpha + \cos \beta &= 2 \cos\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right) \\ \cos \alpha - \cos \beta &= -2 \sin\left(\frac{\alpha + \beta}{2}\right) \sin\left(\frac{\alpha - \beta}{2}\right) \end{aligned}$$

#### Cofunction Formulas

$$\begin{aligned} \sin\left(\frac{\pi}{2} - \theta\right) &= \cos \theta & \cos\left(\frac{\pi}{2} - \theta\right) &= \sin \theta \\ \csc\left(\frac{\pi}{2} - \theta\right) &= \sec \theta & \sec\left(\frac{\pi}{2} - \theta\right) &= \csc \theta \\ \tan\left(\frac{\pi}{2} - \theta\right) &= \cot \theta & \cot\left(\frac{\pi}{2} - \theta\right) &= \tan \theta \end{aligned}$$

**Unit Circle**



For any ordered pair on the unit circle  $(x, y)$  :  $\cos\theta = x$  and  $\sin\theta = y$

**Example**

$$\cos\left(\frac{5\pi}{3}\right) = \frac{1}{2} \quad \sin\left(\frac{5\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

**Inverse Trig Functions**

**Definition**

$y = \sin^{-1} x$  is equivalent to  $x = \sin y$   
 $y = \cos^{-1} x$  is equivalent to  $x = \cos y$   
 $y = \tan^{-1} x$  is equivalent to  $x = \tan y$

**Inverse Properties**

$$\begin{aligned} \cos(\cos^{-1}(x)) &= x & \cos^{-1}(\cos(\theta)) &= \theta \\ \sin(\sin^{-1}(x)) &= x & \sin^{-1}(\sin(\theta)) &= \theta \\ \tan(\tan^{-1}(x)) &= x & \tan^{-1}(\tan(\theta)) &= \theta \end{aligned}$$

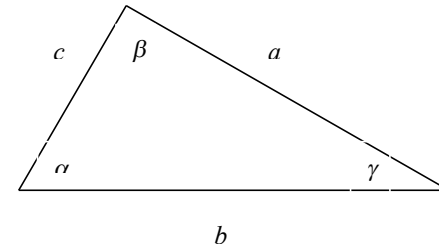
**Domain and Range**

Function	Domain	Range
$y = \sin^{-1} x$	$-1 \leq x \leq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
$y = \cos^{-1} x$	$-1 \leq x \leq 1$	$0 \leq y \leq \pi$
$y = \tan^{-1} x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$

**Alternate Notation**

$$\begin{aligned} \sin^{-1} x &= \arcsin x \\ \cos^{-1} x &= \arccos x \\ \tan^{-1} x &= \arctan x \end{aligned}$$

**Law of Sines, Cosines and Tangents**



**Law of Sines**

$$\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c}$$

**Law of Tangents**

$$\frac{a-b}{a+b} = \frac{\tan \frac{1}{2}(\alpha - \beta)}{\tan \frac{1}{2}(\alpha + \beta)}$$

**Law of Cosines**

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc \cos \alpha \\ b^2 &= a^2 + c^2 - 2ac \cos \beta \\ c^2 &= a^2 + b^2 - 2ab \cos \gamma \end{aligned}$$

$$\frac{b-c}{b+c} = \frac{\tan \frac{1}{2}(\beta - \gamma)}{\tan \frac{1}{2}(\beta + \gamma)}$$

$$\frac{a-c}{a+c} = \frac{\tan \frac{1}{2}(\alpha - \gamma)}{\tan \frac{1}{2}(\alpha + \gamma)}$$

**Mollweide's Formula**

$$\frac{a+b}{c} = \frac{\cos \frac{1}{2}(\alpha - \beta)}{\sin \frac{1}{2}\gamma}$$