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ADVANCE MATHEMATICS

Group-1

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2.1.1 Introduction

The concept of function s is very fundamental in Advance and Modern mathematics. Many mathematicians have contributed to the simple development of idea of function. In 1673, Liebnitz give the definition of function in the context of co-ordinates, the slope of a normal at a point on curve and slope of tangent, as a quantity at every point.

2.1.2 Function

We study a function. It is one of the most important concepts in mathematics.

Definition: Let *A* and *B* be two non-empty sets. If $\exists \; f : A \to B$ such that $f \subset (A \times B)$ and $f \neq \varnothing$. Then $f : A \to B$ is called a function, if $\forall x \in A$, there corresponding a unique ordered pair $\left(x, f(x)\right) \in f$.

The set *A* is called the domain and *B* is called co-domain of *f*. The set $\left\{ y = f(x) \mid (x, y) \in f \right\}$ is called the range of *f*. The domain and range of *f* are denoted by D_f and R_f respectively.

Example-1: Let N be the set of natural numbers. Define a real valued function *f* : N→N by *f* (*x*) = 3*x* + 1. Using this definition, complete the table given below.

x	**1**	**2**	**3**	**4**	**5**	**6**	**7**
f(x)							

Soln:- The complete table is

x	1	2	3	4	5	6	7
f(x)	4	7	10	13	16	19	22

Example-2: Find the range of following functions:

1. $f : N \to R, f(x) = 2x^2 - 1, x \in N$
2. $f : N \to Q, f(x) = \dfrac{1}{x^2} - 1, x \in N$

Sol:- 1. Here, $x \in N$ then $x^2 > 0$

$\therefore\ 2x^2 - 1 > -1$, for any $x \in N$

$\therefore\ f(x) > -1$

Thus $R_f = (-1, \infty)$.

2. Here, $x \in N$ then $\frac{1}{x^2} > 0$

$\therefore\ \frac{1}{x^2} - 1 > -1$

$\therefore\ f(x) > -1$

Thus $R_f = (-1, \infty)$.

2.1.3 Type of some functions and properties:

1. **One-one:** If $f : A \to B$ be any function. Then *f* is said to be one-one function. If $\exists\ \forall a, b \in A$ such that $f(a) = f(b)$ then *a=b.*
2. **Many-one function:** A function *f* is said to be many one function, if *f* is not one-one function.
3. **Onto function:** A function *f* is called onto function, each elements of codomain is the image of some element of domain.
4. **Identity function:** A function $I : A \to A$ is said to be identity finction, if *I(x)=x*, for any $x \in A$.
5. **Constant function:** A function *f* is said to be constant function, if *f(x)=c*, for c is constant.
6. **Bijective function:** Let *f* be any one-one and onto function. Then *f* is said to be bijective function.

2.1.4 Algebra of functions: Let $f : A \to B$ and $g : C \to D$ then

1. **Addition/Subtraction of two functions:** $(f \pm g)(x) = f(x) \pm g(x)$, for all *x*.
2. **Multiplication by scalar:** *f(ax)=af(x).*
3. **Multiplication of two functions: (***fg)(x)=f(x)g(x).*
4. **Division of two functions:** $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$
5. **Composition of function:** $(f \circ g)(x) = f\left(g(x)\right)$ or $(g \circ f)(x) = g\left(f(x)\right)$.
6. **Inverse function:** A function f^{-1} is said to be inverse function of a function *f*, if $(f^{-1} \circ f)(x) = I = (f \circ f^{-1})(x)$.

Example-:3 If *f(x)*=x^2 and *g(x)=2x+1*then find (*f+g)(x), (f-g)(x), fg(x),* $f^{-1}(x)$, $g^{-1}(x)$
$\left(\frac{f}{g}\right)(x),\ (f \circ g)(x),\ (g \circ f)(x),$.

Soln:- Here $f(x)=x^2$ and *g(x)=2x+1.*

Now, $(f+g)(x)=f(x)+g(x)=x^2+2x+1$,

$(f-g)(x)=f(x)-g(x)=x^2-2x-1$,

$fg(x)=f(x)g(x)=x^2(2x+1)=2x^3+x$,

$\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}=\frac{x^2}{2x+1}$,

$(f\circ g)(x)=f(g(x))=f(2x+1)=(2x+1)^2=4x^2+4x+1$,

$(g\circ f)(x)=g(f(x))=f(x^2)=2x^2+1$

And suppose $f(x)=x^2=y$

then $x=\sqrt{y}$. So, $f^{-1}(x)=\sqrt{x}$.

Now suppose $g(x)=2x+1=z$.

Then $x=\frac{z-1}{2}$. So, $g^{-1}(x)=\frac{x-1}{2}$.

Example-:4 If $f(x)=\log_2 x$ and $g(x)=x^4$, then prove that $(f\circ g)(2)=4$. (June 2009)

Soln:- Here $g(x)=x^4$

$\therefore\ g(2)=2^4=16$

Now $f(x)=\log_2 x$

$(f\circ g)(2)=f(g(2))=f(16)=\log_2 16=\log_2 2^4=4(1)=4$

Example-:5 If $f(x)=2x+1$ and $g(x)=x^2-2$, then find

i.) g∘f

ii.)f∘g .

Soln:- *i.)* $(g\circ f)(x)=g(f(x))$

$=g(t)$ where suppose $t=f(x)$

$=t^2-2=(2x+1)^2-2=4x^2+4x-1$

ii.) $(f\circ g)(x)=f(g(x))$

$=f(w)$ where $w=g(x)$

$=2w+1=2(x^2-2)+1=2x^2-3$

Example-:6 If $f:Z\to R$ **be any function is defined by** $f(x)=\frac{3x+2}{5}$.

Soln:- Suppose $y=\frac{3x+2}{5}$.

$\therefore 5y=3x+2$

$\therefore 5y-2=3x$

$\therefore \frac{5y-2}{3}=x$

$\therefore\ f^{-1}(x)=\frac{5x-2}{3}$

Example-:7 If $f(x)=\frac{1+x}{1-x}$, then prove that $f\left(\frac{x+y}{1+xy}\right)=f(x)f(y)$. (June 2013)

Soln:- $1+\frac{x+y}{1+xy}=\frac{1+xy+x+y}{1+xy}=\frac{1(1+x)+y(1+x)}{1+xy}=\frac{(1+x)(1+y)}{1+xy}$

Llly, $1-\frac{x+y}{1+xy}=\frac{(1-x)(1-y)}{1+xy}$ also $f(x)=\frac{1+x}{1-x}$.

L.H.S.= $f\left(\frac{x+y}{1+xy}\right)=\frac{1+\frac{x+y}{1+xy}}{1-\frac{x+y}{1-xy}}=\frac{\frac{(1+x)(1+y)}{1+xy}}{\frac{(1-x)(1-y)}{1+xy}}$

$$=\frac{(1+x)(1+y)}{(1-x)(1-y)}=\left(\frac{1+x}{1-x}\right)\left(\frac{1+y}{1-y}\right)=f(x)f(y)$$

=R.H.S

Example-:8 Prove that $f:R\to R$ **, *f(x)=ax+b* is bijective.**

Soln:- Let $f(x)=f(y)$

$\therefore$ $ax+b=ay+b$

$\therefore$ $ax=by$

$\therefore$ $x=y$

$\therefore$ f is one-one function.

Let $y=ax+b \Leftrightarrow x=\frac{y-b}{a}$.

$$\Leftrightarrow f(x)=f\left(\frac{y-b}{a}\right)=a\left(\frac{y-b}{a}\right)+b=y-b+b=y$$

$\therefore$ f is onto

$\therefore$ f is bijection map.

Example-:9 If $f(x)=\frac{1-x}{1+x}$**, prove that**

i) $f(x)=f^{-1}(x)$

ii) $f(x)+f\left(\frac{1}{x}\right)=0$ **(July 2010,2014)**

iii) $f(x)-f\left(\frac{1}{x}\right)=2f(x)$ **(Dec. 2008, June2014)**

iv) $f(x)f(-x)=1$ **(June 2011,2013)**

Soln:- i) $(f\circ f)(x)=f(f(x))$

$$=f\left(\frac{1-x}{1+x}\right)=\frac{1-\frac{1-x}{1+x}}{1+\frac{1-x}{1+x}}$$

$$=\frac{1+x-1+x}{1+x+1-x}=x$$

$\Rightarrow (f\circ f)(x)=x$

$\Rightarrow f(x) = f^{-1}(x)$

ii) L.H.S= $f(x) + f\left(\frac{1}{x}\right)$

$$= \frac{1-x}{1+x} + \frac{1-\frac{1}{x}}{1+\frac{1}{x}} = \frac{1-x}{1+x} + \frac{x-1}{x+1}$$

$$= \frac{1-x+x-1}{1+x} = 0 = \textbf{R.H.S}$$

iii) L.H.S= $f(x) - f\left(\frac{1}{x}\right)$

$$= \frac{1-x}{1+x} - \frac{x-1}{x+1} = \frac{1-x}{1+x} + \frac{1-x}{1+x}$$

$$= 2\frac{1-x}{1+x} = 2f(x) = \text{R.H.S}$$

iv) L.H.S $= f(x)f(-x) = \frac{1-x}{1+x} \cdot \frac{1-(-x)}{1+(-x)} = \frac{1-x}{1+x} \cdot \frac{1+x}{1-x} = 1 =$**R.H.S**

Example-:10 If *f(x)*=log *x*, prove that

i). *f(xy)=f(x)+f(y)* and

ii). *f(x/y)=f(x)-f(y)*. (jan. 2012, Dec. 2013)

Soln:- Here *f(x)*=log*x*

i) *f(xy)*=log *xy*=log *x*+log *y*=*f(x)*+*f(y)*

ii) *f(x/y)*=log *(x/y)*=log *x*-log *y*=*f(x)*-*f(y)*.

Example-:11 If $f(x) = \log\left(\frac{x}{x-1}\right)$**, prove that** $f(a+1) + f(a) = \log\left(\frac{a+1}{a-1}\right)$. **(june 2014)**

Soln:- Here $f(x) = \log\left(\frac{x}{x-1}\right)$ then $f(a) = \log\left(\frac{a}{a-1}\right)$ and $f(a+1) = \log\left(\frac{a+1}{a+1-1}\right) = \log\left(\frac{a+1}{a}\right)$

L.H.S.= $f(a+1) + f(a) = f(a+1) = \log\left(\frac{a+1}{a}\right) + \log\left(\frac{a}{a-1}\right)$

$$= \log\left(\frac{a+1}{a}\frac{a}{a-1}\right) = \log\left(\frac{a+1}{a-1}\right) = \text{R.H.S}$$

Example-:12 If $f(x) = \log\left(\frac{1-x}{1+x}\right)$**, then prove that** $f\left(\frac{2x}{1+x^2}\right) = 2f(x)$ **.(Dec. 2014)**

Soln:- Here $f(x) = \log\left(\frac{1-x}{1+x}\right)$, now suppose that $y = \frac{2x}{1+x^2}$

Now, L.H.S $= f\left(\frac{2x}{1+x^2}\right)$

$= f(y)$

$= \log\left(\frac{1-y}{1+y}\right)$

$= \log\left(\frac{1-\frac{2x}{1+x^2}}{1+\frac{2x}{1+x^2}}\right)$

$= \log\left(\frac{1+x^2-2x}{1+x^2+2x}\right)$

$= \log\frac{(1-x)^2}{(1+x)^2}$

$= \log\left(\frac{(1-x)}{(1+x)}\right)^2$

$= 2\log\left(\frac{1-x}{1+x}\right)$

$= 2f(x)$

= R.H.S

Example-:13 If $f(x) = \frac{x+3}{4x-5}$ **and** $t = \frac{3+5x}{4x-1}$**, then prove that** $x = f(t)$ **.(June 2014)**

Soln:- Here, $t = \frac{3+5x}{4x-1}$

R.H.S. $= f(t)$

$= \frac{t+3}{4t-5}$

$= \frac{\frac{3+5x}{4x-1}+3}{4\frac{3+5x}{4x-1}-5}$

$= \frac{3+5x+12x-3}{12+20x-20x+5}$

$= \frac{17x}{17}$

$= x$

=L.H.S.

Exercise 2.1.1

Find the f^{-1} ,if it is exists(1 to 5)

1. $f : R \to R$, $f(x) = 2x+5$

2. $f: R \to R$, $f(x) = \frac{6x+5}{7}$

3. $f: N \to N$ $f(x) = x^3$

4. $f: N \to N$, $f(x) = 7x+2$

5. $f: R \to R$, $f(x) = \sqrt{x^3}$

6. **If** $f(x) = \frac{ax+b}{bx+a}$**, then prove that** $f(x)f(1/x) = 1$.

7. **If** $f(x) = e^x$**, prove that** ***f(xy)=f(x)+f(y).***

8. **If** ***f(x)=tan x*****, then prove that i)** $f(x+y) = \frac{f(x)+f(y)}{1-f(x)f(y)}$ **and ii)** $f(2x) = \frac{2f(x)}{1-[f(x)]^2}$

9. **If** $f(x) = \log\left(\frac{x-1}{x}\right)$**, then prove that** $f(x)+f(-x)=f(x^2)$

10. **If** $f(x) = \log\left(\frac{x-1}{x}\right)$**, then is it bijective or not?**

11. **If** $f(x) = \frac{1+x}{1-x}$**, then prove that** $x(f(x))+1=0$**. (June 2014)**

2.2.1 Introduction

This is an introduction to Calculus. Calculus is that branch of mathematics which many deals with the study of change in the value of a function as the points in the domain change. Then we give a definition of limit and study some algebraic limits.

2.2.2 Definition: If *f(x)* becomes arbitrarily close to a unique number *L* as approaches from either side, then the **limit** of *f(x)* as approaches is *L*. This is written as $\lim_{x\to c} f(x) = L$

"If $\exists$ $\lim_{x\to a-} f(x)$ and $\lim_{x\to a+} f(x)$ are exists such that both are equals, then $\lim_{x\to a} f(x)$ is exists with $\lim_{x\to a} f(x) = \lim_{x\to a+} f(x) = \lim_{x\to a-} f(x)$."

Example-1: Verify: $\lim_{x\to \frac{1}{2}} \frac{4x^2-1}{2x-1} = 2$ by following table. Where $x \neq \frac{1}{2}$.

X	049	0.499	0.4999	0.51	0.501	0.5001
f(x)	1.98	1.998	1.9998	2.02	2.002	2.0002

Here, $f(x) = \frac{4x^2-1}{2x-1} = 2x+1$, where $x \neq \frac{1}{2}$.

Then we know that $\lim_{x\to \frac{1}{2}} f(x) = 2$.

Example-2: Find $\lim_{x\to 0} |x|$.

Soln:- We know that $f(x) = \begin{cases} x & x \geq 0 \\ -x & x \leq 0 \end{cases}$

Then

x	-0.1	-0.01	-0.001	0.1	0.01	0.001
f(x)	0.1	0.01	0.001	0.1	0.01	0.001

Then we say that $\lim_{x\to 0} f(x) = 0$

Example-3: Prove that $\lim_{x\to 2} [x]$ is not exists.

Soln:- $f(x) = \begin{cases} 1 & 1 \leq x < 2 \\ 2 & 2 \leq x < 3 \end{cases}$

X	1.9	1.99	1.999	1.9999	2.1	2.01	2.001	2.0001
f(x)	1	1	1	1	2	2	2	2

Then $\lim_{x\to 2-} f(x)=1$ and $\lim_{x\to 2+} f(x)=2$

Then $\lim_{x\to 2+} f(x)=2\neq 1=\lim_{x\to 2-} f(x)=1$

Then $\lim_{x\to 2} f(x)$ is not possible.

2.2.3 Properties of limits:

Let $\lim_{x\to c} f(x)=L$ and $\lim_{x\to c} g(x)=K$ then

1. **Scalar Multiplication:** $\lim_{x\to c}\left[bf(x)\right]=bL$
2. **Addition and subtraction:** $\lim_{x\to c}\left[f(x)\pm g(x)\right]=L\pm K$
3. **Product:** $\lim_{x\to c}\left[f(x)\cdot g(x)\right]=L\cdot K$
4. **Quotient:** $\lim_{x\to c}\frac{f(x)}{g(x)}=\frac{L}{K}$
5. **Power:** $\lim_{x\to c}\left[f(x)\right]^n=\left[\lim_{x\to c} f(x)\right]^n=L^n$

Examples-4: Prove that, if *f(x)* is constant function and *f(x)=c* , *then* $\lim_{x\to a} f(x)=c$ or in other word $\lim_{x\to a} c=c$ and if $\lim_{x\to a} f(x)$ is exists then $\lim_{x\to a} cf(x)=c\lim_{x\to a} f(x)$.

Soln:- Suppose *f(x)=c* and $x\in(a-¢,a+¢)-\{a\}$. Suppose *l=c.*

$a-¢<x<a+¢$, $x\neq a\Rightarrow|f(x)-l|=|c-c|<£$ where $0<£$.

$\therefore \lim_{x\to a} f(x)=c$

$\therefore \lim_{x\to a} c=c$

If $\lim_{x\to a} f(x)$ is exist then $\lim_{x\to a} cf(x)=\lim_{x\to a} c\lim_{x\to a} f(x)=c\lim_{x\to a} f(x)$.

Example-5: Find $\lim_{x\to 2}\left(2x^3+3x^2-5x+1\right)$.

Soln:- Here, $\lim_{x\to 2}\left(2x^3+3x^2-5x+1\right)=2\cdot 2^3+3\cdot 2^2-5\cdot 2+1$

$=16+12-10+1=19$

Example-6: Find $\lim\limits_{x\to 1}\dfrac{x^2+1}{x^2+3x+4}$.

Soln:- Here for $x=1, x^2+3x+4\neq 0$

$$\lim_{x\to 1}\frac{x^2+1}{x^2+3x+4}=\frac{1+1}{1+3+4}=\frac{2}{8}=\frac{1}{4}$$

Example-7: Find $\lim\limits_{x\to 0}\dfrac{x^3-3x^2+x}{4x^3-5x^2+3x}$.

Soln:- $\lim\limits_{x\to 0}\dfrac{x^3-3x^2+x}{4x^3-5x^2+3x}=\lim\limits_{x\to 0}\dfrac{x\left(x^2-3x+1\right)}{x\left(4x^2-5x^2+3\right)}$

$$=\lim_{x\to 0}\frac{x^2-3x+1}{4x^2-5x^2+3}$$

$$=\frac{1}{3}$$

Example-8: Find $\lim\limits_{x\to 1}\dfrac{x^4-7x^3+8x^2-3x+1}{3x^4-5x^3+6x^2-10x+6}$.

Soln:- $\lim\limits_{x\to 1}\dfrac{x^4-7x^3+8x^2-3x+1}{3x^4-5x^3+6x^2-10x+6}=\lim\limits_{x\to 1}\dfrac{(x-1)\left(x^3-6x^2+2x-1\right)}{(x-1)\left(3x^3-2x^2+4x-6\right)}$

$$=\lim_{x\to 1}\frac{x^3-6x^2+2x-1}{3x^3-2x^2+4x-6}$$

$$=\frac{-4}{-1}=4$$

Example-9: Find $\lim\limits_{x\to 2}\dfrac{x^3-5x^2+8x-4}{2x^3-9x^2+12x-4}$.

Soln:- Suppose $p(x)=x^3-5x^2+8x-4$ and $q(x)=2x^3-9x^2+12x-4$.

If $x=2$, then $p(2)=8-20+16-4=0$ and $q(2)=16-36+24-4=0$.

$\therefore (x-2)$ is co-factor of *p(x)* and *q(x)*.

$$\lim_{x\to 2}\frac{x^3-5x^2+8x-4}{2x^3-9x^2+12x-4}=\lim_{x\to 2}\frac{(x-2)^2(x-1)}{(x-2)^2(2x-1)}$$

$$= \lim_{x \to 2} \frac{x-1}{2x-1} = \frac{1}{3}$$

Example-10: Find $\lim_{x \to 1} \frac{x^3 - 3x^2 + 3x - 1}{x^2 - 1}$.

Soln:- $\lim_{x \to 1} \frac{x^3 - 3x^2 + 3x - 1}{x^2 - 1} = \lim_{x \to 1} \frac{(x-1)^3}{(x-1)(x+1)}$

$$= \lim_{x \to 1} \frac{(x-1)^2}{(x+1)} = \frac{0}{2} = 0.$$

Example-11: Find $\lim_{x \to 0} \frac{x^4 - x^3 + x^2}{x^6 - x^5 + x}$.

Soln:- $\lim_{x \to 0} \frac{x^4 - x^3 + x^2}{x^6 - x^5 + x} = \lim_{x \to 0} \frac{x^2(x^2 - x + 1)}{x(x^5 - x^4 + 1)}$

$$= \lim_{x \to 0} \frac{x(x^2 - x + 1)}{x^5 - x^4 + 1} = \frac{0(1)}{1} = 0.$$

2.2.4 Some important formulae of limits:

1. $\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$; for $\forall a \in R$
2. $\lim_{x \to 0} \cos x = 1$
3. $\lim_{x \to 0} \frac{\sin x}{x} = 1$
4. $\lim_{x \to 0} \frac{\tan x}{x} = 1$
5. $\lim_{x \to a} \sin x = \sin a$
6. $\lim_{x \to a} \cos x = \cos a$
7. $\lim_{x \to 0} \frac{\sin^{-1} x}{x} = 1$
8. $\lim_{x \to 0} \frac{\tan^{-1} x}{x} = 1$

Example-12: Find $\lim_{x \to 1} \frac{x^{18} - 1}{x^{16} - 1}$. **(Jan.2012)**

Soln:- $\lim_{x \to 1} \frac{x^{18} - 1}{x^{16} - 1} = \lim_{x \to 1} \frac{x^{18} - 1}{x - 1} \times \frac{x - 1}{x^{16} - 1}$

$$=\frac{\lim_{x\to 1}\frac{x^{18}-1}{x-1}}{\lim_{x\to 1}\frac{x^{16}-1}{x-1}}=\frac{18(1)^{17}}{16(1)^{15}}=\frac{18}{16}=\frac{9}{8}$$

Example-13: Find $\lim_{x\to -2}\frac{x^5+32}{x^3+8}$.

Soln:- $\lim_{x\to -2}\frac{x^5+32}{x^3+8}=\lim_{x\to -2}\frac{x^5-(-2)^5}{x^3-(-2)^3}$

$$=\frac{\lim_{x\to -2}\frac{x^5-(-2)^5}{x-(-2)}}{\lim_{x\to -2}\frac{x^3-(-2)^3}{x-(-2)}}=\frac{5(-2)^4}{3(-2)^2}=\frac{20}{3}$$

Example-14: Find $\lim_{x\to 2}\frac{x^4-16}{x^3-3x^2+3x-2}$.

Soln:- $\lim_{x\to 2}\frac{x^4-16}{x^3-3x^2+3x-2}$

$$=\frac{\lim_{x\to 2}\frac{x^4-2^4}{x-2}}{\lim_{x\to 2}\frac{(x-2)(x^2-x+1)}{x-2}}=\frac{4(2)^3}{4-2+1}=\frac{32}{3}.$$

Example-15: Find $\lim_{x\to 0}\frac{(x+2)^5-32}{x}$.

Soln:- $\lim_{x\to 0}\frac{(x+2)^5-32}{x}$

$$=\lim_{x\to 0}\frac{(x+2)^5-(2)^5}{(x+2)-2}=\lim_{x+2\to 2}\frac{(x+2)^5-(2)^5}{(x+2)-2}$$

$$=5(2)^4=5(16)=80$$

Example-16: Find $\lim_{x\to 0}\frac{\sqrt{25+x}-5}{x}$ **(June-2013)**

Soln:- $\lim_{x\to 0}\frac{\sqrt{25+x}-5}{x}=\lim_{x\to 0}\frac{(25+x)^{1/2}-(25)^{1/2}}{(25+x)-25}$

$$=\frac{1}{2}(25)^{\frac{1}{2}-1}=\frac{1}{2(25)^{1/2}}=\frac{1}{10}$$

Example-17: Find $\lim_{x\to 2}\frac{x^3-8}{\sqrt{x^2+x+2}-\sqrt{3x+2}}$

Soln:- $\lim_{x\to 2}\frac{x^3-8}{\sqrt{x^2+x+2}-\sqrt{3x+2}}$

$$=\lim_{x\to 2}\frac{(x-2)(x^2+2x+4)}{\sqrt{x^2+x+2}-\sqrt{3x+2}}\times\frac{\sqrt{x^2+x+2}+\sqrt{3x+2}}{\sqrt{x^2+x+2}+\sqrt{3x+2}}=\lim_{x\to 2}\frac{(x-2)(x^2+2x+4)\left(\sqrt{x^2+x+2}+\sqrt{3x+2}\right)}{(x^2+x+2)-(3x+2)}$$

$$=\lim_{x\to 2}\frac{(x-2)(x^2+2x+4)\left(\sqrt{x^2+x+2}+\sqrt{3x+2}\right)}{x^2-2x}=\lim_{x\to 2}\frac{(x-2)(x^2+2x+4)\left(\sqrt{x^2+x+2}+\sqrt{3x+2}\right)}{x(x-2)}$$

$$=\lim_{x\to 2}\frac{(x^2+2x+4)\left(\sqrt{x^2+x+2}+\sqrt{3x+2}\right)}{x}=\frac{(12)(\sqrt{8}+\sqrt{8})}{2}=6(4\sqrt{2})=24\sqrt{2}$$

Example-18: Find $\lim_{x\to 0}\frac{\sin ax}{\sin bx}$. $a,b\neq 0$

Soln:- $\lim_{x\to 0}\frac{\sin ax}{\sin bx}$

$$=\lim_{x\to 0}\frac{\frac{\sin ax}{ax}\cdot a}{\frac{\sin bx}{bx}\cdot b}=\frac{\lim_{x\to 0}\frac{\sin ax}{ax}\cdot a}{\lim_{x\to 0}\frac{\sin bx}{bx}\cdot b}=\frac{a\cdot 1}{b\cdot 1}=\frac{a}{b}$$

Example-19: Find $\lim_{x\to 0}\frac{1-\cos x}{x^2}$.

Soln:- $\lim_{x\to 0}\frac{1-\cos x}{x^2}$

$$=\lim_{x\to 0}\frac{(1-\cos x)(1+\cos x)}{x^2(1+\cos x)}=\lim_{x\to 0}\frac{1-\cos^2 x}{x^2(1+\cos x)}$$

$$=\lim_{x\to 0}\frac{\sin^2 x}{x^2}\lim_{x\to 0}\frac{1}{1+\cos x}=1.\frac{1}{1+1}=1/2$$

Example-20: Find $\lim_{x\to 0}\frac{1-\cos mx}{1-\cos nx}$.

Soln:- $\lim\limits_{x\to 0}\dfrac{1-\cos mx}{1-\cos nx}$

$$=\lim_{x\to 0}\frac{\dfrac{1-\cos mx}{(mx)^2}.m^2}{\dfrac{1-\cos nx}{(nx)^2}.n^2}=\frac{\lim\limits_{x\to 0}\dfrac{1-\cos mx}{(mx)^2}.m^2}{\lim\limits_{x\to 0}\dfrac{1-\cos nx}{(nx)^2}.n^2}$$

$$=\frac{\dfrac{m^2}{2}}{\dfrac{n^2}{2}}=\frac{m^2}{n^2}$$

Example-21: Find $\lim\limits_{x\to 0}\dfrac{\sin ax+bx}{ax+\sin bx}$.

Soln:- $\lim\limits_{x\to 0}\dfrac{\sin ax+bx}{ax+\sin bx}$

$$=\lim_{x\to 0}\frac{\dfrac{\sin ax}{x}+bx}{ax+\dfrac{\sin bx}{x}}=\lim_{x\to 0}\frac{a\dfrac{\sin ax}{ax}+bx}{ax+\dfrac{\sin bx}{bx}b}$$

$$=\frac{a+b}{a+b}=1$$

Example-22: Find $\lim\limits_{x\to 0}\dfrac{\tan 5x-3x}{4x-\sin 2x}$

Soln:- $\lim\limits_{x\to 0}\dfrac{\tan 5x-3x}{4x-\sin 2x}$

$$=\lim_{x\to 0}\frac{5\dfrac{\tan 5x}{5x}-3}{4-2\dfrac{\sin 2x}{2x}}=\frac{5-3}{4-2}=1$$

Example-23: Find $\lim\limits_{x\to\frac{\text{®}}{2}}\left(\sec x-\tan x\right)$.

Soln:- $\lim\limits_{x\to\frac{\text{®}}{2}}\left(\sec x-\tan x\right)=\lim\limits_{x\to\frac{\text{®}}{2}}\left(\dfrac{1}{\cos x}-\dfrac{\sin x}{\cos x}\right)$

$$=\lim_{x\to\frac{®}{2}}\left(\frac{1-\sin x}{\cos x}\right)=\lim_{x\to\frac{®}{2}}\frac{(1-\sin x)(1+\sin x)}{\cos x(1+\sin x)}$$

$$=\lim_{x\to\frac{®}{2}}\frac{1-\sin^2 x}{\cos x(1+\sin x)}$$

$$=\lim_{x\to\frac{®}{2}}\frac{\cos x}{1+\sin x}$$

$$=\frac{\cos\frac{®}{2}}{1+\sin\frac{®}{2}}=\frac{0}{2}=0$$

Example-24: Find $\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}$

Soln:- $\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}$

$$=\lim_{x\to 0}\frac{\frac{\sin x}{\cos x}-\sin x}{x^3}\quad=\lim_{x\to 0}\frac{\sin x-\sin x\cos x}{x^3\cos x}$$

$$=\lim_{x\to 0}\frac{\sin x(1-\cos x)}{x^3\cos x}$$

$$=\lim_{x\to 0}\frac{\tan x}{x}\frac{(1-\cos x)}{x^2}$$

$$=1\left(\frac{1}{2}\right)=\frac{1}{2}$$

Example-25: Find $\lim_{x\to\frac{®}{4}}\frac{\sin x-\cos x}{x-\frac{®}{4}}$

Soln:- $\lim_{x\to\frac{®}{4}}\frac{\sin x-\cos x}{x-\frac{®}{4}}$

$$= \lim_{x \to \frac{®}{4}} \frac{\sqrt{2}\left(\frac{1}{\sqrt{2}}\sin x - \frac{1}{\sqrt{2}}\cos x\right)}{x - \frac{®}{4}}$$

$$= \lim_{x \to \frac{®}{4}} \frac{\sqrt{2}\left(\cos\frac{®}{4}\sin x - \sin\frac{®}{4}\cos x\right)}{x - \frac{®}{4}}$$

$$= \lim_{x \to \frac{®}{4}} \frac{\sqrt{2}\sin\left(x - \frac{®}{4}\right)}{x - \frac{®}{4}} = \sqrt{2}$$

Example-26: Find $\lim_{x \to 1}\left(\frac{1}{x-1} - \frac{2}{x^2-1}\right)$.

Soln:- $\lim_{x \to 1}\left(\frac{1}{x-1} - \frac{2}{x^2-1}\right)$

$$= \lim_{x \to 1}\left(\frac{x+1-2}{x^2-1}\right)$$

$$= \lim_{x \to 1}\left(\frac{x-1}{(x-1)(x+1)}\right)$$

$$= \lim_{x \to 1}\left(\frac{1}{x+1}\right) = 1/2$$

Example-27: Find $\lim_{x \to \frac{1}{\sqrt{2}}} \frac{x - \cos(\sin^{-1} x)}{1 - \tan(\sin^{-1} x)}$.

Soln:- $\lim_{x \to \frac{1}{\sqrt{2}}} \frac{x - \cos(\sin^{-1} x)}{1 - \tan(\sin^{-1} x)}$

$$= \lim_{x \to \frac{1}{\sqrt{2}}} \frac{x - \cos(\sin^{-1} x)}{1 - \frac{\sin(\sin^{-1} x)}{\cos(\sin^{-1} x)}} = \lim_{x \to \frac{1}{\sqrt{2}}} \frac{x - \cos(\sin^{-1} x)}{1 - \frac{x}{\cos(\sin^{-1} x)}} = \lim_{x \to \frac{1}{\sqrt{2}}} (-\cos(\sin^{-1} x))$$

$$= -\frac{1}{\sqrt{2}}$$

Example-28: Find $\lim_{x\to 0}\frac{\sin 7x-\sin 3x}{\sin x}$ **.(June 2013)**

Soln:- $\lim_{x\to 0}\frac{\sin 7x-\sin 3x}{\sin x}$

$$=\lim_{x\to 0}\frac{\frac{\sin 7x}{x}-\frac{\sin 3x}{x}}{\frac{\sin x}{x}}$$

$$=\lim_{x\to 0}\frac{\frac{\sin 7x}{7x}7-\frac{\sin 3x}{3x}3}{\frac{\sin x}{x}}$$

$$=\frac{7-3}{1}=4$$

Example-:29 Find $\lim_{x\to 0}\frac{\tan 5x}{\sin 3x}$ **.(June 2013)**

Soln:- $\lim_{x\to 0}\frac{\tan 5x}{\sin 3x}=\lim_{x\to 0}\frac{\frac{\tan 5x}{5x}5}{\frac{\sin 3x}{3x}3}=\frac{5}{3}$

Example-30: Find $\lim_{x\to 1}\frac{x^2-4x+3}{x^2+2x-3}$ (**Dec.2013**)

Soln:- $\lim_{x\to 1}\frac{x^2-4x+3}{x^2+2x-3}=\lim_{x\to 1}\frac{(x-3)(x-1)}{(x+3)(x-1)}$

$$=\lim_{x\to 1}\frac{x-3}{x+3}=\frac{1-3}{1+3}=-\frac{2}{4}=-1/2$$

Example-:31 Find $\lim_{x\to 0}\frac{\cos ecx-\cot x}{x}$ (**Dec.2013**)

Soln:- $\lim_{x\to 0}\frac{\cos ecx-\cot x}{x}$

$$=\lim_{x\to 0}\frac{\frac{1}{\sin x}-\frac{\cos x}{\sin x}}{x}$$

$$=\lim_{x\to 0}\frac{1-\cos x}{x\sin x}=\lim_{x\to 0}\frac{1-\cos x}{x^2}\cdot\frac{x}{\sin x}$$

$$=\frac{1}{2}$$

Exercise 2.2.1

Find the limit of followings:

1. $\lim\limits_{x\to 2}\frac{x^3-3x^2+5x-6}{x^3-8}=\frac{5}{12}$

2. $\lim\limits_{x\to 5}\frac{x^2-9x+20}{x^2-6x+5}=1/4$

3. $\lim\limits_{x\to 1}\frac{(2x-3)(\sqrt{x}-1)}{x^2+x-2}=-1/6$

4. $\lim\limits_{x\to 0}\frac{\sqrt{x+a}-\sqrt{a}}{x\sqrt{a+x}}=1/2a$

5. $\lim\limits_{x\to\frac{\circledR}{2}}\frac{\cot x}{\frac{\circledR}{2}-x}=1$

6. $\lim\limits_{x\to 0}\frac{2\sin x-\sin 2x}{x^3}=1$

7. $\lim\limits_{x\to 0}\frac{1-\cos x\sqrt{\cos 2x}}{x^2}=3/2$

8. $\lim\limits_{x\to 2}\frac{x^2-4}{\sqrt{x+2}-\sqrt{3x-2}}=-8$

9. $\lim\limits_{x\to 2}\frac{x^4-3x^3+2}{x^3-5x^2+3x+1}=5/4$

10. $\lim\limits_{x\to 1}\frac{5x^4+3x^3+x^2-2x-7}{x^4+5x^3+x^2-6x-1}=29/15$

11. $\lim\limits_{x\to\frac{\circledR}{3}}\frac{\sqrt{3}\cos x-\sin x}{x-\frac{\circledR}{3}}=-2$

12. $\lim\limits_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}=\frac{1}{2\sqrt{x}}$

13. $\lim\limits_{x\to a}\frac{x^{2n}-a^{2n}}{x^2-a^2}=na^{2n-1}$

14. $\lim\limits_{x\to 1}\frac{1-x^{-\frac{1}{7}}}{1-x^{-\frac{1}{2}}}=2/7$

15. $\lim\limits_{x\to 1}\frac{\sqrt{x-1}}{\sqrt{x^2-1}+\sqrt{x^3-1}}=\sqrt{3}-\sqrt{2}$

16. $\lim\limits_{x\to a}\frac{\sqrt{2a+3x}-\sqrt{x+4a}}{\sqrt{3a+2x}+\sqrt{4x+a}}=-1$

17. $\lim\limits_{x\to 0}\frac{\sqrt{x^2+x+1}-\sqrt{x+1}}{x^2}=1/2$

Note:-

1. $\lim\limits_{n\to\infty}\frac{1}{n}=0$
2. $\lim\limits_{n\to\infty}r^n=0$
3. $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e$
4. $\lim\limits_{n\to 0}\frac{a^n-1}{n}=\log_e a$
5. $\lim\limits_{n\to 0}\frac{e^n-1}{n}=\log_e e=1$

Example-:32 Find $\lim\limits_{x\to 0}\frac{e^{3x}-1}{x}$.

Soln:- $\lim\limits_{x\to 0}\frac{e^{3x}-1}{x}=\lim\limits_{x\to 0}3\left(\frac{e^{3x}-1}{3x}\right)=3$

Example-:33 Find $\lim\limits_{x\to 0}\frac{e^{ax}-e^{bx}}{x}$

Soln:- $\lim\limits_{x\to 0}\frac{e^{ax}-e^{bx}}{x}=\lim\limits_{x\to 0}\frac{e^{ax}-1+1-e^{bx}}{x}$

$$=\lim_{x\to 0}\frac{e^{ax}-1}{x}-\lim_{x\to 0}\frac{e^{bx}-1}{x}$$

$$=\lim_{x\to 0}a\frac{e^{ax}-1}{ax}-\lim_{x\to 0}b\frac{e^{bx}-1}{bx}=a-b$$

Example-:34 Find $\lim\limits_{x\to 0}\left(\frac{e^x+\sin 2x-1}{x}\right)$**.(June 2013)**

Soln:- $\lim\limits_{x\to 0}\left(\frac{e^x+\sin 2x-1}{x}\right)$

$$=\lim_{x\to 0}\left(\frac{e^x-1+\sin 2x}{x}\right)=\lim_{x\to 0}\frac{e^x-1}{x}+\lim_{x\to 0}2\frac{\sin 2x}{2x}=1+2=3$$

Example-:35 Find $\lim_{x\to 0}\frac{e^x - e^{\tan x}}{x-\tan x}$

Soln:- If $x \to 0$ then $x - \tan x \to 0$.

$$\lim_{x\to 0}\frac{e^x - e^{\tan x}}{x-\tan x} = \lim_{x\to 0} e^{\tan x}\left(\frac{e^{x-\tan x}-1}{x-\tan x}\right) = e^0(1) = 1$$

Example-:36 Find $\lim_{x\to 0}\frac{2^{x+5}-32}{x}$

Soln:- $\lim_{x\to 0}\frac{2^{x+5}-32}{x} = \lim_{x\to 0}\frac{2^x 2^5 - 2^5}{x}$

$$= 32\lim_{x\to 0}\frac{2^x-1}{x} = 32\log_e 2$$

Example-:37 Find $\lim_{x\to 1}\frac{\log x}{1-x}$.

Soln:- Suppose $logx=y$ then $x=e^y$. Moreover $x \to 1$ then $y \to 0$.

$$\lim_{x\to 1}\frac{\log x}{1-x} = \lim_{y\to 0}\frac{y}{1-e^y} = - = \lim_{y\to 0}\frac{y}{e^y-1} = -1$$

Example-:38 Find $\lim_{x\to e^3}\frac{\log x - 3}{x-e^3}$

Soln:- Suppose $logx=y$ then $x=e^y$. Moreover $x \to e^3$ then $y \to 3$.

$$\lim_{x\to e^3}\frac{\log x-3}{x-e^3} = \lim_{y\to 3}\frac{y-3}{e^y-e^3}$$

$$= \lim_{y\to 3}\frac{y-3}{e^3(e^{y-3}-1)} = \frac{1}{e^3}$$

Example-:39 Find $\lim_{x\to 0}\frac{\log(a+x)-\log(a-x)}{x}, (a>0)$

Soln:- $\lim_{x\to 0}\frac{\log(a+x)-\log(a-x)}{x}$

$$= \lim_{x\to 0}\frac{1}{x}\log\left(\frac{a+x}{a-x}\right)$$

$$=\lim_{x\to 0}\log\left(\frac{a+x}{a-x}\right)^{\frac{1}{x}}$$

$$=\log\lim_{x\to 0}\left(\frac{a+x}{a-x}\right)^{\frac{1}{x}}=\log\left(\frac{a\lim_{x\to 0}\left\{1+\frac{x}{a}\right\}}{a\lim_{x\to 0}\left\{1-\frac{x}{a}\right\}}\right)^{\frac{1}{x}}$$

$$=\log\left(\frac{\lim_{x\to 0}\left\{\left(1+\frac{x}{a}\right)^{\frac{a}{x}}\right\}^{\frac{1}{a}}}{\lim_{x\to 0}\left\{\left(1-\frac{x}{a}\right)^{-\frac{a}{x}}\right\}^{-\frac{1}{a}}}\right)=\log\left(\frac{e^{\frac{1}{a}}}{e^{-\frac{1}{a}}}\right)=\log e^{\frac{2}{a}}=\frac{2}{a}$$

Example-:40 Find $\lim_{x\to 0}\left(\frac{1-2x}{1+2x}\right)^{\frac{1}{3x}}$

Soln:- $\lim_{x\to 0}\left(\frac{1-2x}{1+2x}\right)^{\frac{1}{3x}}$

$$=\lim_{x\to 0}\frac{(1-2x)^{\frac{1}{3x}}}{(1+2x)^{\frac{1}{3x}}}=\frac{\lim_{x\to 0}(1-2x)^{\frac{1}{3x}}}{\lim_{x\to 0}(1+2x)^{\frac{1}{3x}}}$$

$$=\frac{\lim_{x\to 0}\left[(1-2x)^{-\frac{1}{2x}}\right]^{-\frac{2}{3}}}{\lim_{x\to 0}\left[(1+2x)^{\frac{1}{2x}}\right]^{\frac{2}{3}}}$$

$$=\frac{e^{-\frac{2}{3}}}{e^{\frac{2}{3}}}=\frac{1}{e^{\frac{4}{3}}}$$

Example-:41 Find $\lim_{x\to 0}\frac{10^x-5^x-2^x+1}{x\sin x}$

Soln:- $\lim_{x\to 0}\frac{10^x-5^x-2^x+1}{x\sin x}$

$$= \lim_{x \to 0} \frac{5^x 2^x - 5^x - 2^x + 1}{x \sin x}$$

$$= \lim_{x \to 0} \frac{\left(5^x - 1\right)\left(2^x - 1\right)}{x \cdot x} \frac{x}{\sin x}$$

$$= \lim_{x \to 0} \frac{\left(5^x - 1\right)}{x} \lim_{x \to 0} \frac{\left(2^x - 1\right)}{x} \lim_{x \to 0} \frac{x}{\sin x}$$

$$= \log 5 \cdot \log 2 \cdot 1 = \log 5 \cdot \log 2$$

Example-:42 Find $\lim_{x \to 0} \frac{15^x - 5^x - 3^x + 1}{x^2}$ **(June 2013)**

Soln:- $\lim_{x \to 0} \frac{15^x - 5^x - 3^x + 1}{x^2}$

$$= \lim_{x \to 0} \frac{5^x 3^x - 5^x - 3^x + 1}{x \cdot x}$$

$$= \lim_{x \to 0} \frac{\left(5^x - 1\right)\left(3^x - 1\right)}{x \cdot x}$$

$$= \lim_{x \to 0} \frac{\left(5^x - 1\right)}{x} \lim_{x \to 0} \frac{\left(3^x - 1\right)}{x}$$

$$= \log 5 \cdot \log 3$$

Example-:43 Find $\lim_{x \to 0} \frac{x \log(1+x)}{1 - \cos x}$ **(july 2014)**

Soln:- $\lim_{x \to 0} \frac{x \log(1+x)}{1 - \cos x}$

$$= \lim_{x \to 0} \frac{\log(1+x)}{x} \frac{1}{\frac{1 - \cos x}{x^2}}$$

$$= \lim_{x \to 0} \frac{\log(1+x)}{x} \cdot \frac{1}{\lim_{x \to 0} \frac{1 - \cos x}{x^2}}$$

$$= \lim_{x \to 0} \frac{1}{x} \log(1+x) \cdot \frac{1}{\frac{1}{2}}$$

$$= 2\lim_{x\to 0}\log(1+x)^{\frac{1}{x}}$$

$$= 2\log\lim_{x\to 0}(1+x)^{\frac{1}{x}}$$

$$= 2\log e = 2$$

Example-:44 Find $\lim_{n\to\infty}\frac{2n-1}{3n+5}$.

Soln:- $\lim_{n\to\infty}\frac{2n-1}{3n+5} = \lim_{n\to\infty}\frac{2-\frac{1}{n}}{3+\frac{5}{n}}$

$$= \frac{2-0}{3+0} = \frac{2}{3}$$

Example-:45 Find $\lim_{n\to\infty}\frac{5n^3-7n^2+3n-1000}{8n^3+18n^2-6n+2004}$

Soln:- $\lim_{n\to\infty}\frac{5n^3-7n^2+3n-1000}{8n^3+18n^2-6n+2004}$

$$= \lim_{n\to\infty}\frac{n^3\left(5-\frac{7}{n}+\frac{3}{n^2}-\frac{1000}{n^3}\right)}{n^3\left(8+\frac{18}{n}-\frac{6}{n^2}+\frac{2004}{n^3}\right)}$$

$$= \frac{5-0+0-0}{8+0-0+0} = 5/8$$

Example-:46 Prove that $\lim_{n\to\infty}\frac{(-1)^n}{n} = 0$

Proof:- Here we know that $\left|\frac{(-1)^n}{n}\right| = \frac{1}{n}$ and

$\lim_{n\to\infty}\frac{1}{n} = 0$. So, $\lim_{n\to\infty}\frac{(-1)^n}{n} = 0$.

Example-:47 Find $\lim_{n\to\infty}\left(\sqrt{n^2+n+1}-n\right)$. **(July 2014)**

Soln:- $\lim_{n\to\infty}\left(\sqrt{n^2+n+1}-n\right)$

$$=\lim_{n\to\infty}\frac{\left(\sqrt{n^2+n+1}-n\right)\left(\sqrt{n^2+n+1}+n\right)}{\left(\sqrt{n^2+n+1}+n\right)}$$

$$=\lim_{n\to\infty}\frac{n^2+n+1-n^2}{\left(\sqrt{n^2+n+1}+n\right)}$$

$$=\lim_{n\to\infty}\frac{n+1}{\sqrt{n\left(1+n+\frac{1}{n^2}\right)}+n}$$

$$=\lim_{n\to\infty}\frac{n\left(1+\frac{1}{n}\right)}{n\left\{\sqrt{\left(1+n+\frac{1}{n^2}\right)}+1\right\}}$$

$$=\lim_{n\to\infty}\frac{1+\frac{1}{n}}{\sqrt{\left(1+n+\frac{1}{n^2}\right)}+1}=1/2$$

Exercise 2.2.2

Find the limit of followings:

1. $\lim_{x\to 0}\frac{a^x-a^{-x}}{x}$

2. $\lim_{x\to 0}\frac{x}{e^{\frac{1}{x}}-1}$

3. $\lim_{x\to 0}\frac{\log(1+3x)}{4^x-1}$

4. $\lim_{x\to 0}\frac{\log(1+x)}{x}$

5. $\lim_{x\to 0}\frac{5^{3x}-1}{3^{5x}-1}$

6. $\lim_{x\to 0}\frac{a^x-b^x}{x}$

7. $\lim_{x\to 0}\frac{e^{5x}-1}{\sin x}$

8. $\lim_{x\to 0}\frac{\log(x+a)-\log a}{x}$

9. $\lim_{x\to 0}\frac{x(e^x-1)}{1-\cos x}$

10. $\lim_{x\to 0}\frac{e^{x+1}-\sin x-e^4}{x}$

Note

UNIT

Three

Differentiation and its Applications

UNIT-3.1

3.1.13 Differentiation

3.1.1 Introduction

We have seen in the unit 3, that by knowing the position of a body at various time intervals it is possible to find the rate at which the position of the body is changing. It is of very general interest to know a certain parameter at various instants of time and try to finding the rate at which it is changing. There are several real life situations where such a process needs to be carried out. For instance, people maintaining a reservoir need to know when will a reservoir overflow knowing the depth of the water at several instances of time, Rocket Scientists need to compute the precise velocity with which the satellite needs to be shot out from the rocket knowing the height of the rocket at various times. Financial institutions need to predict the changes in the value of a particular stock knowing its present value. In these, and many such cases it is desirable to know how a particular parameter is changing with respect to some other parameter. The heart of the matter is derivative of a function at a given point in its domain of definition.

3.1.2 **Definition:-** Suppose f is a real valued function and a is a point in its domain of definition. The derivative of f at a is defined by

$$\frac{d}{dx} f(x) = f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Example 1: Find the derivatives of $f(x)=x^n$, $n \in R$

Here $f(x)=x^n$

Then $f(x+h)=(x+h)^n$

Now, $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^n - x^n}{(x+h) - x}$

Now we take $x+h=t$, then $f'(x) = \lim_{t \to x} \frac{t^n - x^n}{t - x} = nx^{n-1}$

Thus, $\frac{d}{dx}(x^n) = nx^{n-1}$

Example 2: Find the derivatives of *f(x)*=sin *x*. (Nov. 2002)

Here ***f(x)***=sin *x, then f(x+h)*=sin (*x+h)*

Then *f(x+h)-f(x)*=sin (*x+h)-sin x=2 cos(x+h/2) sin(h/2)*

Now $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$$f'(x) = \lim_{h\to 0} \frac{2\cos\left(x+\frac{h}{2}\right)\sin\frac{h}{2}}{h} = \lim_{h\to 0} \frac{\sin\frac{h}{2}}{\frac{h}{2}} \bullet \lim_{h\to 0} \cos\left(x+\frac{h}{2}\right) = \cos x$$

Thus, $\frac{d}{dx}\sin x = \cos x$.

Example 3: Let $f: R \to R$, $f(x) = 3x + 5$. If $f'(1)$ is exists then find it.

$$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h\to 0} \frac{3(1+h) + 5 - 8}{h} = \lim_{h\to 0} \frac{3h}{h} = 3$$

Therefore *f'(1)* is exists and *f'*(1)=3.

Example 4: Let $f: R \to R$, *f(x)=sin x*, if *f'(0)* exists then find it.

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{\sinh - 0}{h} = \lim_{h\to 0} \frac{\sinh}{h} = 1$$

Therefore *f'(0)* is exists and *f'*(0)=1.

Exercise 3.1.1

Find the followings

1. $\frac{d}{dx}\cos x = -\sin x$
2. $\frac{d}{dx}\tan x = -\sec^2 x$
3. $\frac{d}{dx}e^x = e^x$
4. $\frac{d}{dx}a^x = a^x \log_e a$
5. $\frac{d}{dx}\log x = \frac{1}{x}$
6. $\frac{d}{dx}10x$
7. $\frac{d}{dx}2\sin^2 x + 3\cos x + 1$
8. $\frac{d}{dx}\frac{1-\cos x}{\sin x}$
9. $\frac{d}{dx}\sin^4 x$

3.1.3 Important formulae for derivatives:

No.	*f(x)*	*f'(x)*	Remarks
1.	*c*(constant)	0	
2.	x^n	nx^{n-1}	$n \in R$
3.	log *x*	$\frac{1}{x}$	$x \neq 0$
4.	a^x	$a^x \log_e a$	
5.	e^x	e^x	
6.	sin *x*	cos *x*	
7.	cos *x*	-sin *x*	
8.	tan *x*	$\sec^2 x$	
9.	cosec *x*	-cosec *x* cot *x*	
10.	sec *x*	sec *x* tan *x*	
11.	cot *x*	$-\text{cosec}^2 x$	
12.	$\frac{1}{x}$	$-\frac{1}{x^2}$	$x \neq 0$
13.	$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$	$x \neq 0$
14.	$\sin^{-1}x$	$\frac{1}{\sqrt{1-x^2}}$	$\|x\|<1$
15.	$\cos^{-1}x$	$-\frac{1}{\sqrt{1-x^2}}$	$\|x\|<1$
16.	$\tan^{-1}x$	$\frac{1}{1+x^2}$	
17.	$\cot^{-1}x$	$-\frac{1}{1+x^2}$	
18.	$\sec^{-1}x$	$\frac{1}{\|x\|\sqrt{x^2-1}}$	$x \neq 1$
19.	$\text{cosec}^{-1}x$	$-\frac{1}{\|x\|\sqrt{x^2-1}}$	$x \neq 1$

3.1.4 Working rules of Differentiation:

If *f* and *g* be two functions of *x*, such that

1. **Derivative of sum and difference:** If $y = u \pm v$, then $\frac{dy}{dx} = \frac{du}{dx} \pm \frac{dv}{dx}$.
2. **Derivative of product:** If *y=uv, then* $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$.

3. If $y = uvw$, then $\frac{dy}{dx} = uv\frac{dw}{dx} + vw\frac{du}{dx} + wu\frac{dv}{dx}$.

4. If $y = ku$, then $\frac{dy}{dx} = k\frac{du}{dx}$, where k is constant.

5. **Derivatives of division:** If $y = \frac{u}{v}$, then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$.

6. If $y = \frac{k}{v}$, then $\frac{dy}{dx} = -\frac{k}{u^2}\frac{du}{dx}$.

Example 5: **Find** $\frac{d}{dx}\left(x^3 + 3^x + 3^3\right)$**.**

$$\frac{d}{dx}\left(x^3 + 3^x + 3^3\right) = \frac{d}{dx}x^3 + \frac{d}{dx}3^x + \frac{d}{dx}3^3$$

$$= 3x^2 + 3^x \log 3 + 0 = 3x^2 + 3^x \log 3$$

Example 6: **Find** $\frac{d}{dx}\left(x^{\frac{3}{2}} + \frac{1}{x^{\frac{5}{2}}} + \left(\sqrt{2}\right)^x\right)$**.**

$$\frac{d}{dx}\left(x^{\frac{3}{2}} + \frac{1}{x^{\frac{5}{2}}} + \left(\sqrt{2}\right)^x\right) = \frac{d}{dx}\left(x^{\frac{3}{2}} + x^{-\frac{5}{2}} + \left(\sqrt{2}\right)^x\right)$$

$$= \frac{3}{2}x^{\frac{1}{2}} - \frac{5}{2}x^{-\frac{7}{2}} + \left(\sqrt{2}\right)^x \log 2^{1/2}$$

$$= \frac{3}{2}\sqrt{x} - \frac{5}{2x^{\frac{7}{2}}} + \frac{1}{2}2^{\frac{1}{2}x} \log 2$$

Example 7: **Find** $\frac{d}{dx}\sqrt[5]{x}$**.**

$$\frac{d}{dx}\sqrt[5]{x} = \frac{d}{dx}x^{\frac{1}{5}} = \frac{1}{5}x^{\frac{1}{5}-1} = \frac{1}{x}x^{-\frac{4}{5}}$$

Example 8: **Find** $\frac{d}{dx}\left(x - \frac{1}{x}\right)^2$**.**

$$\frac{d}{dx}\left(x - \frac{1}{x}\right)^2 = \frac{d}{dx}\left(x^2 - 2 + \frac{1}{x^2}\right) = \frac{d}{dx}\left(x^2 - 2 + x^{-2}\right) = 2x - 0 - 2x^{-3} = 2x - 2x^{-3}$$

Example 9: Find $\frac{d}{dx}\left(a^x + \log x + \frac{1}{x}\right)$.

$$\frac{d}{dx}\left(a^x + \log x + \frac{1}{x}\right) = \frac{d}{dx}a^x + \frac{d}{dx}\log x + \frac{d}{dx}x^{-1} = a^x \log a + \frac{1}{x} - \frac{1}{x^2}$$

Example 10: Find $\frac{d}{dx}\left(x^4 - \frac{2}{x^2}\right)\left(x^2 - \frac{3}{x^3}\right)$.

$$\frac{d}{dx}\left(x^4 - \frac{2}{x^2}\right)\left(x^2 - \frac{3}{x^3}\right) = \frac{d}{dx}\left(x^6 - 3x - 2 + 6x^{-5}\right)$$

$$= 6x^5 - 3 - 30x^{-6}$$

Example 11: Find $\frac{d}{dx}\frac{4x^2 - 5x + 1}{x^5 - x^4}$.

$$\frac{d}{dx}\frac{4x^2 - 5x + 1}{x^5 - x^4} = \frac{d}{dx}\frac{(x-1)(4x-1)}{x^4(x-1)}$$

$$= \frac{d}{dx}\frac{4x-1}{x^4} = \frac{d}{dx}4x^{-3} - \frac{d}{dx}x^{-4} = -12x^{-4} + 4x^{-5}$$

Exercise 3.1.2

Find $\frac{dy}{dx}$ **for the followings:**

1. $y = x^{1/2} + x^{1/3} + x^{1/4}$

2. $y = x^{-1/2} + x^{-1/3} + x^{-1/4}$

3. $y = x - \frac{1}{x}$

4. $y = \sqrt{x} - \frac{1}{\sqrt{x}}$

5. $y = \frac{3x^3 - 7x^2 + 1}{x^4}$

6. $y = x^{-\frac{5}{3}}$

7. $y = e^x + x^e + e^e$

8. $y = \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)\left(x + \frac{1}{x}\right)$

9. $y = \frac{1 - x^5}{1 - x}$

Example 12: Find $\frac{d}{dx}\cos^2 x$.

$$\frac{d}{dx}\cos^2 x = \frac{d}{dx}\cos x \cos x$$

$$= \cos x \frac{d}{dx} \cos x + \cos x \frac{d}{dx} \cos x$$
$$= 2\cos x(-\sin x)$$
$$= -2\sin x \cos x$$
$$= -\sin 2x$$

Example 13: Find $\frac{d}{dx}\sin 2x$**.**

$$\frac{d}{dx}\sin 2x = \frac{d}{dx} 2\sin x \cos x = 2\left[\cos x \frac{d}{dx}\sin x + \sin x \frac{d}{dx}\cos x\right]$$
$$= 2\left[\sin x(-\sin x) + \cos x \cos x\right]$$
$$= 2(\cos^2 x - \sin^2 x)$$
$$= 2\cos 2x$$

Example 14: Find $\frac{d}{dx}\left(\frac{a+b\sin x}{c+d\sin x}\right)$**. Where** $c + d\sin x \neq 0$**.**

$$\frac{d}{dx}\left(\frac{a+b\sin x}{c+d\sin x}\right) = \frac{(c+d\sin x)\frac{d}{dx}(a+b\sin x) - (a+b\sin x)\frac{d}{dx}(c+d\sin x)}{(c+d\sin x)^2}$$
$$= \frac{(c+d\sin x)b\cos x - (a+b\sin x)d\cos x}{(c+d\sin x)^2}$$
$$= \frac{bc\cos x + bd\sin x\cos x - ad\cos x - bd\sin x\cos x}{(c+d\sin x)^2}$$
$$= \frac{(bc-ad)\cos x}{(c+d\sin x)^2}$$

Example 15: Find $\frac{d}{dx}\frac{\cos x}{1+\sin x}$**, where** $\sin x \neq -1$**.**

$$\frac{d}{dx}\frac{\cos x}{1+\sin x} = \frac{(1+\sin x)\frac{d}{dx}\cos x - \cos x\frac{d}{dx}(1+\sin x)}{(1+\sin x)^2}$$

$$= \frac{(1+\sin x)(-\sin x)-\cos x\cos x}{(1+\sin x)^2}$$

$$= \frac{-\sin x-\sin^2 x-\cos^2 x}{(1+\sin x)^2}$$

$$= \frac{-(1+\sin x)}{(1+\sin x)^2}$$

$$= \frac{-1}{1+\sin x}$$

Example 16: Find $\frac{d}{dx}\frac{\sec x-1}{\sec x+1}$.

$$\frac{d}{dx}\frac{\sec x-1}{\sec x+1} = \frac{d}{dx}\frac{1-\cos x}{1+\cos x}$$

$$= \frac{d}{dx}\frac{1-\cos x}{1+\cos x}\frac{1-\cos x}{1-\cos x}$$

$$= \frac{d}{dx}\frac{(1-\cos x)^2}{1-\cos^2 x}$$

$$= \frac{d}{dx}\frac{1-2\cos x+\cos^2 x}{\sin^2 x}$$

$$= \frac{d}{dx}(\cos ec^2 x-2\cos ecx\cot x+\cot^2 x)$$

$$= \frac{d}{dx}\cos ec^2 x-2\frac{d}{dx}\cos ecx\cot x+\frac{d}{dx}\cot^2 x$$

$$= 2\cos ecx(-\cos ecx\cot x)-2\left[\cot x(-\cos ecx\cot x)+\cos ecx(-\cos ec^2 x)\right]+2\cot x(-\cos ec^2 x)$$

$$= -2\cos ec^2 x\cot x-2\left[-\cos ecx\cot^2 x-\cos ec^3 x\right]-2\cos ec^2 x\cot x$$

$$= -4\cos ec^2 x\cot x+2\cos ecx\left(\cot^2 x+\cos ec^2 x\right)$$

Example 17: Find $\frac{d}{dx}\sin\left(\cos\left(\tan\left(x^2+2x+3\right)\right)\right)$.

$$\frac{d}{dx}\sin\left(\cos\left(\tan\left(x^2+2x+3\right)\right)\right)$$

$$= \cos\left(\cos\left(\tan\left(x^2+2x+3\right)\right)\right)\bullet\left(-\sin\left(\tan\left(x^2+2x+3\right)\right)\right)\bullet\left(\sec^2\left(x^2+2x+3\right)\right)\bullet(2x+2)$$

Example 18: Find $\frac{d}{dx}x^3\sin x$.

$$\frac{d}{dx}x^3\sin x = \sin x\frac{d}{dx}x^3+x^3\frac{d}{dx}\sin x$$

$$= \sin x(2x^2) + x^3 \cos x$$
$$= 2x^2 \sin x + x^3 \cos x$$
$$= x^2(x \cos x + \sin x)$$

Example 19: Find $\frac{d}{dx} \log x \cdot \tan x$ **.**

$$\frac{d}{dx} \log x \cdot \tan x = \log x \frac{d}{dx} \tan x + \tan x \frac{d}{dx} \log x$$
$$= \log x \sec^2 x + \tan x \left(\frac{1}{x}\right)$$
$$= \log x \sec^2 x + \frac{1}{x} \tan x$$

Example 20: Find $\frac{d}{dx} \frac{x^2-1}{x^2+1}$**. (June 2011)**

$$\frac{d}{dx} \frac{x^2-1}{x^2+1} = \frac{d}{dx} \frac{(x^2+1)\frac{d}{dx}(x^2-1) - (x^2-1)\frac{d}{dx}(x^2+1)}{(x^2+1)^2}$$
$$= \frac{(x^2+1)2x - (x^2-1)2x}{(x^2+1)^2}$$
$$= \frac{2x(x^2+1-x^2+1)}{(x^2+1)^2}$$
$$= \frac{4x}{(x^2+1)^2}$$

Example 21:

Exercise 3.1.3

Find the $\frac{dy}{dx}$ **for following:**

1. $y = \frac{\sin x + \cos x}{\sin x - \cos x}$
2. $y = \frac{4x + 7\sin x}{5x + 8\cos x}$,(2009)
3. $y = \frac{x}{1 + \cot x}$
4. $y = \sin^2 x$
5. $y = \sin^3 x$
6. $y = x^7 + e^x \log x$
7. $y = e^x 3^x x^3$
8. $y = \frac{2x-3}{3x-2}$
9. $y = \frac{\log x}{x}$

Example 22: Find $\frac{d}{dx}\left\{\left(x+\frac{2}{x+3}\right)\left(\frac{2x-1}{x^2+3x+2}\right)\right\}$**.**

Suppose, $y=\left(x+\frac{2}{x+3}\right)\left(\frac{2x-1}{x^2+3x+2}\right)$

$$=\left(\frac{x^2+3x+2}{x+3}\right)\left(\frac{2x-1}{x^2+3x+2}\right)$$

$$=\frac{2x-1}{x+3}$$

Then $\frac{dy}{dx}=\frac{d}{dx}\frac{2x-1}{x+3}$

$$=\frac{(x+3)\frac{d}{dx}(2x-1)-(2x-1)\frac{d}{dx}(x+1)}{(x+3)^2}$$

$$=\frac{2x+6-2x+1}{(x+3)^2}=\frac{7}{(x+3)^2}$$

Example 23: Find $\frac{dy}{dx}$ **for following:**

1. $y=\log\left(\tan x\right)+\cos x$ **(Jan 2012)**
2. $y=\log\left(x+\sqrt{x^2+a^2}\right)$**,(June 2011)**
3. $y=\log\sqrt{\frac{1+\sin x}{1-\sin x}}$ **,(2005)**
4. $y=\log\left(\sin x\right)$**, (Dec.2014)**

Soln:- 1) Here, $y=\log\left(\tan x\right)+\cos x$.

$$\frac{dy}{dx}=\frac{d}{dx}\left(\log(\tan x)\right)+\frac{d}{dx}\cos x$$

$$=\frac{1}{\tan x}\frac{d}{dx}\tan x-\sin x$$

$$=\frac{\cos x}{\sin x}\sec^2 x-\sin x$$

$$=\frac{\cos x}{\sin x}\frac{1}{\cos^2 x}-\sin x=\frac{1}{\sin x\cos x}-\sin x$$

$$= \frac{2}{2\sin x\cos x} - \sin x = \frac{2}{\sin 2x} - \sin x$$
$$= 2\cos ecx - \sin x$$

2) Here, $y = \log\left(x + \sqrt{x^2 + a^2}\right)$

$$\frac{dy}{dx} = \frac{1}{x + \sqrt{x^2 + a^2}} \frac{d}{dx}\left(x + \sqrt{x^2 + a^2}\right) = \frac{1}{x + \sqrt{x^2 + a^2}}\left(\frac{d}{dx}x + \frac{d}{dx}\sqrt{x^2 + a^2}\right)$$
$$= \frac{1}{x + \sqrt{x^2 + a^2}}\left(1 + \frac{1}{2\sqrt{x^2 + a^2}}\frac{d}{dx}\left(x^2 + a^2\right)\right)$$
$$= \frac{1}{x + \sqrt{x^2 + a^2}}\left(1 + \frac{1}{2\sqrt{x^2 + a^2}}(2x)\right)$$
$$= \frac{1}{x + \sqrt{x^2 + a^2}}\left(1 + \frac{x}{\sqrt{x^2 + a^2}}\right) = \frac{1}{x + \sqrt{x^2 + a^2}}\left(\frac{\sqrt{x^2 + a^2} + x}{\sqrt{x^2 + a^2}}\right)$$
$$= \frac{1}{\sqrt{x^2 + a^2}}$$

3) Here $y = \log\sqrt{\frac{1+\sin x}{1-\sin x}} = \frac{1}{2}\log\left(\frac{1+\sin x}{1-\sin x}\right)$

Now, $\frac{1+\sin x}{1-\sin x} = \frac{1+\sin x}{1-\sin x}\frac{1+\sin x}{1+\sin x} = \frac{(1+\sin x)^2}{1-\sin^2 x} = \frac{(1+\sin x)^2}{\cos^2 x} = \left(\frac{1+\sin x}{\cos x}\right)^2$

$$\therefore y = \log\sqrt{\frac{1+\sin x}{1-\sin x}} = \frac{1}{2}\log\left(\frac{1+\sin x}{1-\sin x}\right) = \frac{1}{2}\log\left(\frac{1+\sin x}{\cos x}\right)^2$$
$$= \frac{2}{2}\log\left(\frac{1+\sin x}{\cos x}\right) = \log\left(\frac{1+\sin x}{\cos x}\right) = \log\left(1 + six\right) - \log\left(\cos x\right)$$

Now,

$$\frac{dy}{dx} = \frac{d}{dx}\log\sqrt{\frac{1+\sin x}{1-\sin x}} = \frac{d}{dx}\left(\log(1+\sin x) - \log(\cos x)\right)$$
$$= \frac{d}{dx}\log(1+\sin x) - \frac{d}{dx}\log(\cos x) = \frac{1}{1+\sin x}\frac{d}{dx}\left(1+\sin x\right) - \frac{1}{\cos x}\frac{d}{dx}\cos x$$
$$= \frac{\cos x}{1+\sin x} - \frac{-\sin x}{\cos x} = \frac{\cos x}{1+\sin x}\frac{1-\sin x}{1+\sin x} + \frac{\sin x}{\cos x}$$

$$=\frac{\cos x(1-\sin x)}{\cos^2 x}+\frac{\sin x}{\cos x}$$
$$=\frac{\cos^2 x(1-\sin x)+\sin x\cos^2 x}{\cos^3 x}$$
$$=\frac{1-\sin x+\sin x}{\cos x}=\frac{1}{\cos x}$$
$$=\sec x$$

4). Here, $y=\log\left(\sin x\right)$

$$\therefore \frac{dy}{dx}=\frac{1}{\sin x}\times\frac{d}{dx}\sin x$$
$$\therefore \frac{dy}{dx}=\frac{1}{\sin x}\cos x=\cot x$$

Example 24: If $y=\log\left(\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}-x}\right)$**, then prove that** $\sqrt{x^2+a^2}\,\frac{dy}{dx}=2$ **.(June 2014)**

Soln:- Now, $y=\log\left(\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}-x}\right)$

$$\therefore \frac{dy}{dx}=\frac{1}{\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}-x}}\times\frac{d}{dx}\left(\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}-x}\right)$$

$$\therefore \frac{dy}{dx}=\frac{\sqrt{x^2+a^2}-x}{\sqrt{x^2+a^2}+x}\times\frac{\left(\sqrt{x^2+a^2}-x\right)\left(\frac{1}{2\sqrt{x^2+a^2}}\cdot 2x+1\right)-\left(\sqrt{x^2+a^2}+x\right)\left(\frac{1}{2\sqrt{x^2+a^2}}\cdot 2x-1\right)}{\left(\sqrt{x^2+a^2}-x\right)^2}$$

$$\therefore \frac{dy}{dx}=\frac{\left(\sqrt{x^2+a^2}-x\right)\left(\frac{x}{\sqrt{x^2+a^2}}+1\right)-\left(\sqrt{x^2+a^2}+x\right)\left(\frac{x}{\sqrt{x^2+a^2}}-1\right)}{\left(\sqrt{x^2+a^2}+x\right)\left(\sqrt{x^2+a^2}-x\right)}$$

$$\therefore \frac{dy}{dx}=\frac{\frac{\left(\sqrt{x^2+a^2}-x\right)\left(x+\sqrt{x^2+a^2}\right)-\left(x-\sqrt{x^2+a^2}\right)\left(\sqrt{x^2+a^2}+x\right)}{\sqrt{x^2+a^2}}}{x^2+a^2-x^2}$$

$$\therefore \frac{dy}{dx}=\frac{\left(\sqrt{x^2+a^2}-x\right)\left(x+\sqrt{x^2+a^2}\right)+\left(\sqrt{x^2+a^2}-x\right)\left(\sqrt{x^2+a^2}+x\right)}{a^2\sqrt{x^2+a^2}}$$

$$\therefore \sqrt{x^2+a^2}\frac{dy}{dx}=\frac{x^2+a^2-x^2+x^2+a^2-x^2}{a^2}$$

$$\therefore \sqrt{x^2+a^2}\frac{dy}{dx}=\frac{2a^2}{a^2}$$

$$\therefore \sqrt{x^2+a^2}\frac{dy}{dx}=2$$

Example 25: If $y=\log\left(x+\sqrt{1+x^2}\right)$ **then prove that** $\left(1+x^2\right)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=0$**. (Dec. 2014)**

Soln:- Here, $y=\log\left(x+\sqrt{1+x^2}\right)$

$$\therefore y_1=\frac{1}{x+\sqrt{1+x^2}}\times\frac{d}{dx}\left(x+\sqrt{1+x^2}\right)$$

$$\therefore y_1=\frac{1}{x+\sqrt{1+x^2}}\left(1+\frac{1}{2\sqrt{1+x^2}}2x\right)$$

$$\therefore y_1=\frac{1}{x+\sqrt{1+x^2}}\left(\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}}\right)$$

$$\therefore y_1=\frac{1}{\sqrt{1+x^2}}$$

$$\therefore \sqrt{1+x^2}\,y_1=0$$

$$\therefore \left(1+x^2\right)y_1^{\,2}=0$$

$$\therefore 2\left(1+x^2\right)y_2y_1+2y_1^{\,2}x=0$$

$$\therefore \left(1+x^2\right)y_2+xy_1=0$$

Example 26: If $y=\log\left(\sec x+tenx\right)$ **then find** $\frac{dy}{dx}$**. (Dec.2014)**

Soln:- Here, $y=\log\left(\sec x+tenx\right)$

$$\therefore \frac{dy}{dx}=\frac{1}{\sec x+tenx}\left(\sec x\cdot tenx+\sec^2 x\right)$$

$$\therefore \frac{dy}{dx}=\frac{\sec x\left(\sec x+tenx\right)}{\sec x+tenx}$$

$$\therefore \frac{dy}{dx}=\sec x$$

Example 27: Find $\frac{dy}{dx}$ for followings:

1. $y=\sin^{-1}\frac{x}{a}$

2. $y=\cos^{-1}\left(4x^3-3x\right),\quad 0<x<\frac{1}{2}$

3. $y=\tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}},\quad \text{®}<x<2\text{®}$

4. $y=\tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$

Soln:- 1.) Here, $y=\sin^{-1}\frac{x}{a}$,

$$\therefore \frac{dy}{dx}=\frac{1}{\sqrt{1-\frac{x^2}{a^2}}}\cdot\frac{d}{dx}\left(\frac{x}{a}\right)$$

[પહલા 'બહાર' ના વિધેય $\sin^{-1}$ નુ વિકલન તના ચલ $\frac{x}{a}$ સપક્સ અન પછા 'અંદર' ના વિધેય $\frac{x}{a}$ ના વિક્લાત ના તાન સાથ ગુનાકર કરત.]

$$\therefore \frac{dy}{dx}=\frac{|a|}{\sqrt{a^2-x^2}}\cdot\frac{1}{a}$$

Then *i*) If $a>0$, then $\frac{dy}{dx}=\frac{a}{\sqrt{a^2-x^2}}\frac{1}{a}=\frac{1}{\sqrt{a^2-x^2}}$

ii). If $a<0$, then $\frac{dy}{dx}=\frac{-a}{\sqrt{a^2-x^2}}\frac{1}{a}=\frac{-1}{\sqrt{a^2-x^2}}$

2.) Here, $y=\cos^{-1}\left(4x^3-3x\right)$

Suppose that $z=\cos^{-1}x$ then $x=\cos z$.

Now, $0<x<\frac{1}{2}$

$\Rightarrow \cos\frac{\text{®}}{2}<\cos y<\cos\frac{\text{®}}{3}\Rightarrow\frac{\text{®}}{3}<y<\frac{\text{®}}{2}$, [કારણ ક *cos* પ્રથમ ચરણમા ઘટતુ વિધય છ.]

$$\Rightarrow \text{®} < 3y < \frac{3\text{®}}{2} \Rightarrow 0 < 3y - \text{®} < \frac{\text{®}}{2}$$

Now,

$$\cos^{-1}\left(4x^3 - 3x\right)$$
$$= \cos^{-1}\left(4\cos^3 y - 3\cos y\right)$$
$$= \cos^{-1}\left(\cos 3y\right)$$
$$= \cos^{-1}\left(-\cos(\text{®} - 3y)\right)$$
$$= \cos^{-1}\left(-\cos(3y - \text{®})\right)$$
$$= \text{®} - \cos^{-1}\left(\cos(3y - \text{®})\right)$$
$$= \text{®} - (3y - \text{®})$$
$$= 2\text{®} - 3y$$
$$= 2\text{®} - 3\cos^{-1} x$$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}\cos^{-1}\left(4x^3 - 3x\right)$$
$$= \frac{d}{dx}\left(2\text{®} - 3\cos^{-1} x\right)$$
$$= 0 - 3\frac{-1}{\sqrt{1-x^2}} = \frac{3}{\sqrt{1-x^2}}$$

3.) Here, $y = \tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}} = \tan^{-1}\left|\tan\frac{x}{2}\right|$

Now,

$$\text{®} < x < 2\text{®}$$
$$\Rightarrow \frac{\text{®}}{2} < \frac{x}{2} < \text{®}$$
$$\Rightarrow \tan\frac{x}{2} < 0$$

Then, $y = \tan^{-1}\left|\tan\frac{x}{2}\right| = \tan^{-1}\left(-\tan\frac{x}{2}\right) = -\tan^{-1}\left(\tan\frac{x}{2}\right) = -\frac{x}{2}$.

$$\frac{dy}{dx} = -\frac{1}{2}$$

4.) Here, $y = \tan^{-1} \dfrac{\sqrt{1+x^2}-1}{x}$

Suppose, if $\exists\ \text{—} = \tan^{-1} x \ \ni\ -\dfrac{\text{®}}{2} < \text{—} < \dfrac{\text{®}}{2}$

$\Rightarrow x = \tan \text{—}$

Then,

$$y = \tan^{-1} \frac{\sqrt{1+x^2}-1}{x} = \tan^{-1} \frac{\sqrt{1+\tan^2 \text{—}}-1}{\tan \text{—}}$$

$$= \tan^{-1} \frac{\sec \text{—} - 1}{\tan \text{—}} = \tan^{-1} \frac{1-\cos \text{—}}{\sin \text{—}}$$

$$= \tan^{-1} \frac{2\sin^2 \frac{\text{—}}{2}}{2\sin \frac{\text{—}}{2} \cos \frac{\text{—}}{2}} = \tan^{-1}\left(\tan \frac{\text{—}}{2} \right)$$

$$= \frac{\text{—}}{2} = \frac{1}{2} \tan^{-1} x$$

Then, $\dfrac{dy}{dx} = \dfrac{1}{2\left(1+x^2\right)}$.

Example 28: If $y = \tan^{-1} \sqrt{\dfrac{1-\cos x}{1+\cos x}}$ **then find** $\dfrac{dy}{dx}$**. (Dec 2003)**

Soln:- Here, $y = \tan^{-1} \sqrt{\dfrac{1-\cos x}{1+\cos x}}$

Now, $\dfrac{1-\cos x}{1+\cos x} = \dfrac{2\sin^2 \frac{x}{2}}{2\cos^2 \frac{x}{2}} = \tan^2 \dfrac{x}{2}$

$\therefore \sqrt{\dfrac{1-\cos x}{1+\cos x}} = \tan \dfrac{x}{2}$

$\therefore y = \tan^{-1}\left(\tan \dfrac{x}{2} \right) = \dfrac{x}{2}$

$\therefore \dfrac{dy}{dx} = \dfrac{1}{2}$

Example 29: **If** $y = \tan^{-1}\frac{2x}{1-x^2}$ **then find** $\frac{dy}{dx}$**.**

Soln:- Here, $y = \tan^{-1}\frac{2x}{1-x^2}$

Take $x = \tan\text{¯}$. Then $\frac{2x}{1-x^2} = \frac{2\tan\text{¯}}{1-\tan^2\text{¯}} = \tan 2\text{¯}$

$\therefore y = \tan^{-1}\left(\tan 2\text{¯}\right) = 2\text{¯} = 2\tan^{-1} x$

$\therefore \frac{dy}{dx} = 2\cdot\frac{1}{1+x^2} = \frac{2}{1+x^2}$

Exercise 3.1.4

Find $\frac{dy}{dx}$ **for followings:**

1. $y = \tan^{-1}\frac{x}{a}$
2. $y = \tan^{-1}\sqrt{\frac{1+\sin x}{1-\sin x}}$, $\frac{\text{®}}{2} < x < \text{®}$
3. $y = \tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$

3.1.5 Differentiation of Implicit functions:

Example 30: **If** $x^3 + y^3 = 3axy$**, then find** $\frac{dy}{dx}$**.**

Soln:- Here, $x^3 + y^3 = 3axy$

$\therefore \frac{d}{dx}\left(x^3 + y^3\right) = \frac{d}{dx}3axy$

$\therefore \frac{d}{dx}x^3 + \frac{d}{dx}y^3 = 3a\frac{d}{dx}xy$

$\therefore 3x^2 + 3y^2\frac{dy}{dx} = 3a\left[x\frac{dy}{dx} + y\cdot 1\right]$

$$\therefore x^2 + y^2 \frac{dy}{dx} = ax\frac{dy}{dx} + ay$$

$$\therefore \left(y^2 - ax\right)\frac{dy}{dx} = ay - x^2$$

$$\therefore \frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}$$

Example 31: If $x + y = \sin xy$ **, then find** $\frac{dy}{dx}$**.**

Soln:- Here, $x + y = \sin xy$

$$\therefore \frac{d}{dx}(x + y) = \frac{d}{dx}\sin xy$$

$$\therefore 1 + \frac{dy}{dx} = \cos xy\left(x\frac{dy}{dx} + y\right)$$

$$\therefore \left(1 - x\cos xy\right)\frac{dy}{dx} = y\cos xy - 1$$

$$\therefore \frac{dy}{dx} = \frac{y\cos xy - 1}{1 - x\cos xy}$$

3.1.6 Differentiation of parametric functions:

Suppose $u = f(x)$ and $v = g(x)$ then for $\frac{du}{dv}$, we follows two steps:

1. First find $\frac{du}{dx}$ and $\frac{dv}{dx}$ from the above functions.
2. Then $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$ gives the derivative of *u* with respect to *v*.

Example 32: If $x = \frac{a\left(1+t^2\right)}{1-t^2}$**,** $y = \frac{2bt}{1-t^2}$ **then find** $\frac{dy}{dx}$**. (June 2009)**

OR

Differentiate $x = \frac{a\left(1+t^2\right)}{1-t^2}$ **w.r.t.** $y = \frac{2bt}{1-t^2}$.

Soln:- Here, $x = a\frac{1+t^2}{1-t^2}$

$$\therefore \frac{dx}{dt} = a\left[\frac{\left(1-t^2\right)2t-\left(1+t^2\right)(-2t)}{\left(1-t^2\right)^2}\right] = a\left[\frac{2t\left(1-t^2+1+t^2\right)}{\left(1-t^2\right)}\right] = \frac{4at}{\left(1-t^2\right)^2}$$

And $\quad y = 2b\frac{t}{1-t^2}$

$$\therefore \frac{dy}{dy} = 2b\left[\frac{\left(1-t^2\right)1-t(-2t)}{\left(1-t^2\right)^2}\right] = \frac{2b\left(1+t^2\right)}{\left(1-t^2\right)^2}$$

Then, $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{2b\left(1+t^2\right)}{\left(1-t^2\right)^2} \times \dfrac{\left(1-t^2\right)^2}{4at} = \dfrac{b\left(1+t^2\right)}{2at}$

Example 33: Find $\frac{dy}{dx}$ for followings:

1. $x = a\sec\bar{\ }$ and $y = b\tan\bar{\ }$ **(Jan 2012)**
2. $x = \cos^{3}\bar{\ }$ and $y = \sin^{3}\bar{\ }$ **(June 2011, 2014)**

Soln:- 1.) Here, $x = a\sec\bar{\ }$

$$\therefore \frac{dx}{d\bar{\ }} = a\sec\bar{\ }\ \tan\bar{\ }$$

And $y = b\tan\bar{\ } \Rightarrow \dfrac{dy}{d\bar{\ }} = b\sec^{2}\bar{\ }$

Thus, $\dfrac{dy}{dx} = \dfrac{\frac{dy}{d\bar{\ }}}{\frac{dx}{d\bar{\ }}} = \dfrac{b\sec^{2}\bar{\ }}{a\sec\bar{\ }\ \tan\bar{\ }} = \dfrac{b}{a}\cos ec\bar{\ }$

2.) Here, $x = \cos^{3}\bar{\ } \Rightarrow \dfrac{dx}{d\bar{\ }} = -3\cos^{2}\bar{\ }\ \sin\bar{\ }$

And $y = \sin^{3}\bar{\ } \Rightarrow \dfrac{dy}{d\bar{\ }} = 3\sin^{2}\bar{\ }\ \cos\bar{\ }$

Thus, $\dfrac{dy}{dx} = \dfrac{\frac{dy}{d\bar{\ }}}{\frac{dx}{d\bar{\ }}} = \dfrac{3\sin^{2}\bar{\ }\ \cos\bar{\ }}{-3\cos^{2}\bar{\ }\ \sin\bar{\ }} = -\tan\bar{\ }$

Exercise 3.1.5

Find $\frac{dy}{dx}$ for followings:

1. $x^2y + y^2x = e^{x+y}$
2. $\sin(x-y)+\cos(x+y)=1$
3. $x\sin y + y\sin x = 0$
4. $x = a\sin^{4}\text{—}$ and $y = b\cos^{4}\text{—}$
5. $x = a\sin\text{—}$ and $y = a\left(1+\cos\text{—}\right)$

3.1.7 Logarithmic Differentiation:

Example 34: Find $\frac{d}{dx}x^{\sin x}$.

Soln:- Suppose $y = x^{\sin x}$

$$\Rightarrow \log y = \log x^{\sin x}$$
$$\Rightarrow \log y = \sin x \log x$$

Now, we differentiate it with respect to *x*.

$$\frac{d}{dx}\log y = \frac{d}{dx}\left(\sin x \log x\right)$$
$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{\sin x}{x} + \cos x \log x$$
$$\Rightarrow \frac{dy}{dx} = y\left(\frac{\sin x}{x} + \cos x \log x\right)$$
$$\Rightarrow \frac{dy}{dx} = x^{\sin x}\left(\frac{\sin x}{x} + \cos x \log x\right)$$

Example 35: If $x^y = e^{x-y}$, then prove that $\frac{dy}{dx} = \frac{\log x}{\left(1+\log x\right)^2}$. (June 2009)

Soln:- Here,

$$x^y = e^{x-y}$$
$$\Rightarrow \log x^y = \log e^{x-y} \Rightarrow y\log x = (x-y)\log e \qquad \ldots\ldots(1)$$
$$\Rightarrow y\log x = x-y$$

Now, we differentiate it with respect to *x*.

$$y\frac{1}{x}+\log x\frac{dy}{dx}=1-\frac{dy}{dx}$$

$$\Rightarrow\left(1+\log x\right)\frac{dy}{dx}=1-\frac{y}{x}$$

$$\Rightarrow\frac{dy}{dx}=\frac{1-\frac{y}{x}}{1+\log x}$$

Now, by (1)

$$y\log x=x-y$$

$$\Rightarrow\log x=\frac{x}{y}-1$$

$$\Rightarrow\frac{x}{y}=1+\log x$$

$$\Rightarrow\frac{y}{x}=\frac{1}{1+\log x}$$

Thus,

$$\frac{dy}{dx}=\frac{1-\frac{y}{x}}{1+\log x}$$

$$\Rightarrow\frac{dy}{dx}=\frac{\log x}{\left(1+\log x\right)^2}$$

Example 36: Find $\frac{dy}{dx}$ for followings:

1. $y=x^x$, **(July 2010, Dec. 2014)**
2. $y=\left(\sin x\right)^{\tan x}$, **(June 2011)**
3. $\log y=x^x\log x$, **(June 2014,Dec.2014)**

Soln:- 1.) Here, $y=x^x$

$$\Rightarrow\log y=\log x^x$$

$$\Rightarrow\log y=x\log x$$

Now, we differentiate it with respect to *x*.

$$\frac{1}{y}\frac{dy}{dx} = \log x + \frac{x}{x}$$

$$\Rightarrow \frac{dy}{dx} = y\left(\log x + \frac{x}{x}\right)$$

$$\Rightarrow \frac{dy}{dx} = x^x\left(\log x + \frac{x}{x}\right)$$

2.) Here, $y = (\sin x)^{\tan x}$

$$\Rightarrow \log y = \log(\sin x)^{\tan x}$$

$$\Rightarrow \log y = \tan x \log \sin x$$

Now, we differentiate it with respect to *x*.

$$\frac{1}{\text{y}}\frac{dy}{dx} = \frac{\tan x}{\sin x}\cos x + \log \sin x\left(\sec^2 x\right)$$

$$\Rightarrow \frac{dy}{dx} = y\left(1 + \sec^2 x \log \sin x\right)$$

$$\Rightarrow \frac{dy}{dx} = (\sin x)^{\tan x}\left(1 + \sec^2 x \log \sin x\right)$$

3.) Here, $\log y = x^x \log x$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \log x \frac{d}{dx} x^x + x^x \frac{1}{x}$$

$$\Rightarrow \frac{dy}{dx} = y\left(x^x (1 + \log x) \log x + x^{x-1}\right)$$

Note: We generally use following notations:

1. $y_1 = \frac{dy}{dx}$
2. $y_2 = \frac{d^2 y}{dx^2}$
3. $\frac{d}{dx} y_1 = y_2$
4. $\frac{d}{dx} y_1^{\,2} = 2 y_1 y_2$
5. $\frac{d}{dx} y^2 = 2 y y_1$

Example 37: If $y = x\log x$ then find y_2. (June 2014)

Soln:- Here, $y = x\log x$

$$\Rightarrow y_1 = \log x + \frac{x}{x}$$
$$\Rightarrow y_1 = 1 + \log x$$
$$\Rightarrow y_2 = \frac{1}{x}$$

Example 38: If $x^3 + y^3 = x^3 y^3$ then prove that $\frac{dy}{dx} - \frac{x^2\left(1-y^3\right)}{y^2\left(x^3-1\right)} = 0$.(June 2014)

Soln:- Here, $x^3 + y^3 = x^3 y^3$

$$\frac{d}{dx}\left(x^3 + y^3\right) = \frac{d}{dx}\left(x^3 y^3\right)$$
$$\Rightarrow 3x^2 + 3y^2\frac{dy}{dx} = 3y^3x^2 + 3y^2x^3\frac{dy}{dx} \Rightarrow x^2 - y^3x^2 = y^2x^3\frac{dy}{dx} - y^2\frac{dy}{dx}$$
$$\Rightarrow x^2\left(1-y^3\right) = y^2\left(x^3-1\right)\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{x^2\left(1-y^3\right)}{y^2\left(x^3-1\right)}$$
$$\Rightarrow \frac{dy}{dx} - \frac{x^2\left(1-y^3\right)}{y^2\left(x^3-1\right)} = 0$$

Example 39: If $y = \log\left(x + \sqrt{1+x^2}\right)$ then prove that $\left(1+x^2\right)y_2 + xy_1 = 0$.(Dec. 2014)

Soln:- Here, $y = \log\left(x + \sqrt{1+x^2}\right)$

$$\Rightarrow y_1 = \frac{1}{x+\sqrt{1+x^2}} \times \left\{1 + \frac{1}{2\sqrt{1+x^2}}2x\right\}$$
$$\Rightarrow y_1 = \frac{1}{x+\sqrt{1+x^2}} \times \left\{\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}}\right\}$$
$$\Rightarrow y_1 = \frac{1}{\sqrt{1+x^2}}$$
$$\Rightarrow \sqrt{1+x^2}\,y_1 = 1$$
$$\Rightarrow \left(1+x^2\right)y_1^{\,2} = 1$$
$$\Rightarrow 2\left(1+x^2\right)y_1 y_2 + 2y_1^{\,2}x = 0$$
$$\Rightarrow \left(1+x^2\right)y_2 + xy_1 = 0$$

Example 40: If $y = 2e^{3x} + 3e^{-2x}$ then prove that $y_2 - y_1 - 6y = 0$.(Sept 2009, July 2010, Dec 2014)

Soln:- Here,

$$y = 2e^{3x} + 3e^{-2x}$$
$$\Rightarrow y_1 = 6e^{3x} - 6e^{-2x}$$
$$\Rightarrow y_2 = 18e^{3x} + 12e^{-2x}$$

Now, **L.H.S.** $= y_2 - y_1 - 6y$

$$= 18e^{3x} + 12e^{-2x} - 6e^{3x} + 6e^{-2x} - 12e^{3x} - 18e^{-2x}$$
$$= 0$$

= **R.H.S.**

Example 41: If $y = ae^{kx} + be^{-kx}$, then prove that $y_2 = k^2 y$. (June 2011)

Soln:- Here,

$$y = ae^{kx} + be^{-kx}$$
$$\Rightarrow y_1 = ake^{kx} - bke^{-kx}$$
$$\Rightarrow y_2 = ak^2e^{kx} + bk^2e^{-kx}$$
$$\Rightarrow y_2 = k^2\left(ae^{kx} + be^{-kx}\right)$$
$$\Rightarrow y_2 = k^2 y$$

Example 42: If $y = a\cos(\log x) + b\sin(\log x)$**, then prove that** $x^2 y_2 + xy_1 + y = 0$**.(Jan 2012)**

Soln:- Here, $y = a\cos(\log x) + b\sin(\log x)$

$$\Rightarrow y_1 = -a\sin(\log x)\frac{1}{x} + b\cos(\log x)\frac{1}{x}$$
$$\Rightarrow xy_1 = b\cos(\log x) - a\sin(\log x)$$
$$\Rightarrow y_1 + xy_2 = -b\sin(\log x)\frac{1}{x} - a\cos(\log x)\frac{1}{x}$$
$$\Rightarrow xy_1 + x^2 y_2 = -\left(b\sin(\log x) + a\cos(\log x)\right)$$
$$\Rightarrow xy_1 + x^2 y_2 = -y$$
$$\Rightarrow xy_1 + x^2 y_2 + y = 0$$

Exercise 3.1.6

1. **If** $y = \sin(\sin x)$, **then prove that** $y_2 + y_1 \tan x + y\cos^2 x = 0$.
2. **If** $y = e^x \sin x$**, then prove that** $y_2 - 2y_1 + 2y = 0$.
3. **If** $y = \log\sin x$**, then prove that** $y_2 + {y_1}^2 + 1 = 0$.
4. **If** $y = A\cos pt + B\sin pt$, **then prove that** $y_2 + p^2 y = 0$.
5. **If** $y = e^{m\tan^{-1} x}$, **then prove that** $(1 + x^2)y_2 + (2x - m)y_1 = 0$.

Note

UNIT-3.2

Applications of Differentiation

In Unit-3.1, we have learnt how to find derivative of composite functions, inverse trigonometric functions, implicit functions, exponential functions and logarithmic functions. In this unit, we will study applications of the derivative in various disciplines, e.g., in engineering, science, social science, and many other fields. For instance, we will learn how the derivative can be used (i) to determine rate of change of quantities, (ii) to find the equations of tangent and normal to a curve at a point, (iii) to find turning points on the graph of a function which in turn will help us to locate points at which largest or smallest value (locally) of a function occurs. We will also use derivative to find intervals on which a function is increasing or decreasing. Finally, we use the derivative to find approximate value of certain quantities.

3.2.1 Velocity:

Velocity is the rate of covering distance in a definite direction.

$$\text{velocity} = \frac{\text{distance}}{\text{time}},$$

$\therefore v = \frac{ds}{dt} = f'(t)$, where v=velocity, ds = distance, dt = time.

Unit of velocity is *m/s*.

3.2.2 Acceleration:

Acceleration is the rate of change of velocity.

$$\text{acceleration} = \frac{\text{velocity}}{\text{time}}$$

$\therefore a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2s}{dt^2} = f''(t)$, where a=acceleration, v, s, t are as above.

Unit of acceleration is m/s^2.

NOTE:

1. When the investigation starts, the point of time is taken as t=0.
2. When a particle changes the direction of motion, its velocity is taken as zero, *i.e.* v=0.

Example 43: The equation of motion of a particle is $s = t^3 - 5t^2 + 3t + 1$. Find the time when the particle changes its direction. (Nov. 2009, 2011)

Soln:- Here, $s = t^3 - 5t^2 + 3t + 1$

$$\therefore v = \frac{ds}{dt} = 3t^2 - 10t + 3 = (t-3)(3t-1)$$

When a particle changes a direction, $v=0$.

$\therefore (t-3)(3t-1)=0$

$\therefore t=3$ sec. OR $t=\frac{1}{3}$ sec.

Example 44: The equation of motion of a particle is $s = t^3 - 3t^2 + 4t + 3$. Find its velocity and acceleration at $t=2$. (Jan. 2012)

Soln:- Here, $s = t^3 - 3t^2 + 4t + 3$

$$\therefore v = \frac{ds}{dt} = 3t^2 - 6t + 4$$

$$\therefore a = \frac{d}{dt}\left(\frac{ds}{dt}\right) = 6t - 6$$

At, t=2, velocity $v=3(2)^2-6(2)+4=4$ *cm/sec*

And acceleration $a=6(2)-6=6$ *cm/sec*^{2}.

Example 45: The equation of motion of a particle is $s = t^3 + 2t^2 - 3t + 5$. Find its velocity and acceleration of the particle at $t=1$ and $t=2$ sec.(June 2007,2014)

Soln:- Here, $s = t^3 + 2t^2 - 3t + 5$

$\therefore$ Velocity $v = \frac{ds}{dt} = 3t^2 + 4t - 3$,

Acceleration $a = \frac{dv}{dt} = 6t + 4$.

Case 1: If $t=1$,

Velocity $v = 3(1)^2 + 4(1) - 3 = 4$ cm/sec

Acceleration $a = 6(1) + 4 = 10$ cm/sec^2

Case 2: If $t=2$,

Velocity $v = 3(2)^2 + 4(2) - 3 = 17$ cm/sec

Acceleration $a = 6(2) + 4 = 16$ cm/sec^2.

Example 46: If the equation of motion of a particle is $s=t^3-6t^2+9t+6$, find its velocity when *t=0*. And also find acceleration when *v*=0. (Sept. 2009)

Soln:- Here, $s=t^3-6t^2+9t+6$.

$\therefore$ velocity

$$v=\frac{ds}{dt}=3t^2-12t+9$$
$$=3\left(t^2-4t+3\right)$$
$$=3(t-1)(t-3)$$

$\therefore$ acceleration $a=6t-12=6(t-2)$,

If *t=0*, $v=3(0-1)(0-3)=9$ cm/sec

And when *v*=0, *0=3(t-1)(t-3)*

$\therefore$ *t=1* or *t=3*

Case 1: If *t*=1, $a=-6$ cm/sec^2

Case 2: If *t*=3, $a=6$ cm/sec^2

Example 47: The equation of motion of a particle is $s=t^3+3t$, *t > 0*. Then

i. Find the velocity and acceleration at *t*=3.

ii. When do velocity and acceleration become equal ? (Sept. 2009)

Soln:- Here, $s=t^3+3t$.

$\therefore$ Velocity $v=\frac{ds}{dt}=3t^2+3$

Acceleration is $a=\frac{dv}{dt}=6t$. Now,

i. At *t* =., velocity $v=3(3)^2+3=27$ units and acceleration is $a=6(3)=18$ units.

ii. If velocity and acceleration are equal, then *v* = *a* gives

$$3t^2+3=6t$$
$$\therefore\ 3t^2-6t+3=0$$
$$\therefore\ t^2-2t+1=0$$
$$\therefore\ \left(t-1\right)^2=0$$
$$\therefore\ t=1$$

Example 48: Equation of motion of a particle is $s = 2t^3 - 3t^2 - 12t + 5$. Find the velocity and acceleration of a particle at *t = 1* sec and *t = 2* sec. (June 2014)

Soln:- Here, $s = 2t^3 - 3t^2 - 12t + 5$

$\therefore$ velocity $v = \frac{ds}{dt} = 6t^2 - 6t - 12$

Acceleration $a = \frac{dv}{dt} = 12t - 6$

Case 1: If $t = 1$ sec., then $v = 6(1)^2 - 6(1) - 12 = -12$ m/s

And $a = 12(1) - 6 = 6$ m/s^2.

Case 2: If $t = 2$ sec., then $v = 6(2)^2 - 6(2) - 12 = 0$

And $a = 12(2) - 6 = 18$ m/s^2.

3.2.3 Maxima and Minima